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Riccati equation and the problem of decoherence

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The block operator matrix theory is used to investigate the problem of a single qubit. We establish a connection between the Riccati operator equation and the possibility of obtaining an exact reduced dynamics for the qubit in question. The model of the half spin particle in the rotating magnetic field coupling with the external environment is discussed. We show that the model defined in such a way can be reduced to a time independent problem. © *2010 American Institute of Physics.* [doi:10.1063/1.3442364]

I. INTRODUCTION

Exactly solvable models of decoherence play an important role both in the theory of open quantum systems and the quantum information theory.[1,2] Unfortunately, most of the models describing decoherence process cannot be solved exactly. However, there is a wide class of models for which an exact reduced dynamics is known.[3] These models deal with the case where the energy transfer between the system and the environment is not present. This phenomenon is known as pure decoherence or dephasing.[4] It has been found that generalizations of dephasing models to the case where energy is exchanged between the system and the environment are straightforward, but for most of these generalizations analytical solutions were not obtained. It is natural to ask why the dephasing models can be solved easily, whereas even the most basic generalizations pose such a difficult task.

In this paper we show that obtaining the exact reduced dynamics of the model of one qubit interacting with the environment is at least as difficult as solving the Riccati operator equation associated with the Hamiltonian defining the model. First we will discuss the procedure allowing one to obtain the density matrix for the system using the block operator matrix (BOM) perspective.

The general form of the Hamiltonian describing the qubit Q coupling with the external environment (heat bath) E can be written as[3]

$$H_{QE} = H_Q \otimes 1_E + 1_Q \otimes H_E + H_{\text{int}}, \tag{1}$$

where H_Q and H_E are the Hamiltonians of the qubit and the environment, respectively, and H_{int} represents the interaction between the systems. The Hamiltonian H_{QE} acts on the space $\mathcal{H}_Q \otimes \mathcal{H}_E$, where $\mathcal{H}_Q$ and $\mathcal{H}_E$ are the Hilbert spaces for the system and the environment, respectively. For most models it is assumed that the initial state of the $Q+E$ system has the form $\rho_{QE}=\rho_Q \otimes \rho_E$, i.e., that there is no correlation between Q and E initially (see Ref. 5 and references therein). Our analysis is free of this assumption. The state of the Q system at any given time t has the form

$$\rho_Q(t) = \text{Tr}_E(U_t \rho_Q \otimes \rho_E U_t^\dagger), \tag{2}$$

where U_t is the evolution operator of the $Q+E$ system and by $\text{Tr}_E(\cdot)$ we denote the partial trace. The $\rho_Q(t)$ is called a reduced dynamics (with respect to the degree of freedom of the environment). From now on the quantity $\rho_Q(t)$ will be referred to as the solution of the model. In the case of

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$\mathcal{H}_Q=\mathbb{C}^2$ and $\mathcal{H}_E=\mathcal{H}$, where $\mathcal{H}$ is an arbitrarily separable Hilbert space (in general dim $\mathcal{H}=\infty$) the following isomorphism holds: $\mathbb{C}^2\otimes\mathcal{H}=\mathcal{H}\oplus\mathcal{H}$. Therefore, any given operator A acting on the $\mathbb{C}^2\otimes\mathcal{H}$ space can be thought of as the 2×2 BOM $[A_{ij}]$, where A_{ij} $(i,j=1,2)$ act on $\mathcal{H}$. Using the above notation the procedure of calculating the partial trace Tr_E may be defined in a very intuitive way, namely,

$$\mathrm{Tr}_E(A)=\begin{bmatrix}\mathrm{Tr}\,A_{11} & \mathrm{Tr}\,A_{12}\\ \mathrm{Tr}\,A_{21} & \mathrm{Tr}\,A_{22}\end{bmatrix}, \tag{3}$$

where $\mathrm{Tr}(\cdot)$ is a trace on $\mathcal{H}$. One can easily see that obtaining the reduced dynamics $\rho_Q(t)$ is very simple. However, Eq. (2) is far less useful than its theoretical simplicity might indicate. The reason is that one cannot determine the exact block operator 2×2 matrix form of the evolution operator U_t of the system $Q+E$. The task becomes even more difficult when the Hamiltonian is time dependent.

There have been few theories resolving the problem of finding reduced dynamics both for time dependent and time independent Hamiltonians.[3,6] Furthermore, a majority of the scientists focus their attention on a numerical method and on perfecting the approximation methods.[7] As a consequence, most of the research on the quantum information theory is based on a numerical rather than an analytical approach. As a result, during the past several years no progress has been made in solving the known models.

The main purpose of this paper is to present an analytical approach. We consider one of the most established and useful models, namely, the spin 1/2 (qubit) in the rotating magnetic field. In the case where no coupling with the external environment is present, an analytical solution can be found in an elegant and simple manner.[8] If the mentioned coupling (modeled by quantum system of infinite number of degree of freedom) is present, however, the exact solution has not been found yet. We will not address this in the current manuscript. However, we show that this model can be effectively reduced to the time independent problem (Sec. II). Moreover, we show that the solution of any given model with a time independent Hamiltonian requires solving the Riccati operator equation associated with the Hamiltonian H defining the problem (Sec. III). In other words, we establish the connection between the problem of decoherence in physics and the mathematical problem of resolving the Riccati operator equation. Furthermore, using the results of Sec. III we discuss the possibility of obtaining an exact solution to the analyzed problem from the set of differential equations on $\mathcal{H}\oplus\mathcal{H}$ (Sec. IV). Finally, in Sec. V we consider a spin-boson model as an example. Finally, in Sec. VI we give a summary of the paper.

II. SPIN HALF IN A ROTATING MAGNETIC FIELD AND IN CONTACT WITH ENVIRONMENT

Let us consider a single qubit in rotating magnetic field interacting with its environment. The qubit-environment time dependent Hamiltonian reads

$$H(t,\beta)=H_Q(t,\beta)\otimes 1_E+I_2\otimes H_E+H_{\mathrm{int}}, \tag{4}$$

where $H_Q(t,\beta)$ and H_E are Hamiltonians of qubit Q and the environment, respectively, and H_{int} represents the interaction between Q and the environment. It is assumed that H_{int} takes the form $f(\sigma_3)\otimes V$, where V is a Hermitian operator acting on $\mathcal{H}_E$ and $f(\sigma_3)$ is an analytic function of σ_3. Hamiltonian $H_Q(t,\beta)$ is given by

$$H_Q(t,\beta)=\beta\sigma_3+\alpha(\sigma_1\cos(\omega t)+\sigma_2\sin(\omega t)), \tag{5}$$

and it represents a spin system in rotating magnetic field $\vec{B}(t)$, where

$$\vec{B}(t)=[B_1\cos(\omega t),B_1\sin(\omega t),B_0]. \tag{6}$$

Here, $\alpha=\frac{1}{2}\omega_1\sim B_1$ and $\beta=\frac{1}{2}\omega_0\sim B_0$, where B_0, B_1 are amplitudes of the magnetic field.[9]

The model described by the Hamiltonian (4) cannot be solved exactly in this general case. By this we mean that the exact reduced dynamics $\rho_Q(t)$ for that model are not known. Let us now focus on another model defined by the Hamiltonian $H(\beta)\equiv H(0,\beta)$, where $H(t,\beta)$ is given by (4).

We will show that if η_t is a solution to the model described by the Hamiltonian (4), and $\rho_t(\beta)$ represents a solution of the model with Hamiltonian $H(\beta)$, then the following holds:

$$\eta_t = V_t\rho_t\left(\beta - \frac{\omega}{2}\right)V_t^\dagger, \tag{7}$$

where

$$V_t = \mathrm{diag}(e^{-i\omega t/2}, e^{i\omega t/2}). \tag{8}$$

From Eqs. (7) and (8) we see that if reduced dynamics $\rho_t(\beta)$ is known, then to obtain the solution of the model of the $H(t,\beta)$ Hamiltonian one needs to introduce an effective parameter $\beta_{\mathrm{eff}} := \beta - \omega/2$, replace β by β_{eff}, and perform a unitary transformation (8). Since the procedure explained above is very simple we can effectively reduce the problem of solving model (4) to one of solving the model $H(\beta)$.

In order to prove Eq. (7) let us note that the Hamiltonian (4) satisfies the following condition ($\hbar=1$):

$$H(t,\beta) = e^{iKt}H(\beta)e^{-iKt}, \tag{9}$$

where $K=-(\omega/2)\sigma_3\otimes 1_E$. This can be easily proven using the Baker–Campbell–Hausdorff formula.[10] As it was shown in Ref. 8, every quantum system with Hamiltonian $H(t,\beta)$ satisfying (9), for some Hermitian operator K, evolves as

$$U_t(\beta) = e^{iKt}e^{-iH_{\mathrm{eff}}(\beta)t}, \quad H_{\mathrm{eff}}(\beta) := H(\beta) + K. \tag{10}$$

Note that in general $[H(\beta),K]\neq 0$ and therefore $[H_{\mathrm{eff}}(\beta),K]\neq 0$. In our case, from Eq. (4) we see that $H(\beta)=(\beta\sigma_3+\alpha\sigma_1)\otimes 1_E$; thus we have

$$H_{\mathrm{eff}}(\beta) = (\beta\sigma_3 + \alpha\sigma_1)\otimes 1_E - \frac{\omega}{2}\sigma_3\otimes 1_E = \left(\left(\beta - \frac{\omega}{2}\right)\sigma_3 + \alpha\sigma_1\right)\otimes 1_E = H\left(\beta - \frac{\omega}{2}\right). \tag{11}$$

From Eqs. (10) and (11) we have

$$U_t(\beta) = e^{iKt}U_t\left(\beta - \frac{\omega}{2}\right), \tag{12}$$

where $U_t(\beta)$ is the evolution operator generated by $H(t,\beta)$. Let $\hat{\rho}_t(\beta)$ and $\hat{\eta}_t$ be a density operator for the closed system $Q+E$ associated with Hamiltonian $H(\beta)$ and $H(t,\beta)$, respectively, in arbitrary time t. Let us also assume that $\hat{\rho}_0(\beta)=\hat{\eta}_0\equiv\hat{\rho}$. Using Eq. (12) one can easily see that

$$\hat{\eta}_t = U_t(\beta)\hat{\rho}U_t^\dagger(\beta) = e^{iKt}U_t\left(\beta - \frac{\omega}{2}\right)\hat{\rho}U_t^\dagger\left(\beta - \frac{\omega}{2}\right)e^{-iKt} = \hat{V}_t\hat{\rho}_t\left(\beta - \frac{\omega}{2}\right)\hat{V}_t^\dagger, \tag{13}$$

where we introduced $\hat{V}_t=e^{iKt}$. To end the proof we will show that if $\hat{A}_1, \hat{A}_2 \in B(\mathcal{H}\oplus\mathcal{H})$ are a 2×2 BOM of the form $\hat{A}_i=A_i\otimes 1_E$ $(i=1,2)$ and $\hat{B}=[\hat{B}_{ij}]\in B(\mathcal{H}\oplus\mathcal{H})$ then

$$\mathrm{Tr}_E(\hat{A}_1\hat{B}\hat{A}_2) = A_1\,\mathrm{Tr}_E(\hat{B})A_2. \tag{14}$$

Equation (14) follows from the linearity of trace Tr operation and definition (3) of partial trace. Note that $\hat{V}_t=V_t\otimes 1_E$, where V_t is given by Eq. (8), thus taking partial trace of Eq. (13) and using (14) we obtain (7) with V_t given by (8).

III. OPERATOR RICCATI EQUATION

So far we have shown that the solution η_t can be easily constructed from $\rho_t(\beta)$. Now, we will pay attention to the possibility of obtaining an exact solution $\rho_t(\beta)$. Let us now assume that $f(\sigma_3)=\sigma_3$ and rewrite Hamiltonian $H(\beta)$ as a BOM (Ref. 11) as

$$H(\beta)=\begin{bmatrix} H_+ +\beta & \alpha \\ \alpha & H_- -\beta \end{bmatrix}, \tag{15}$$

where we have introduced $H_\pm = H_E \pm V$. Since Hamiltonian (15) is time independent, we can write the evolution operator as $U_t=\exp(-iH(\beta)t)$. One can see that the main problem here is to represent U_t as 2×2 BOM.

If $\alpha=0$ this problem is trivial. On the other hand, for $\alpha\neq 0$ the diagonalization of 2×2 BOM is required, which is not a trivial problem.[12] With every Hermitian 2×2 BOM of the form

$$R=\begin{bmatrix} A & B \\ B^\dagger & C \end{bmatrix}, \quad A,B,C \in \mathcal{H}, \tag{16}$$

we can associate an *operator Riccati equation*[13]

$$XBX + XA - CX - B^\dagger = 0, \tag{17}$$

where $X\in\mathcal{H}$. Solution X of Eq. (17), if it exists, can be used to construct 2×2 BOM,

$$U_X=\begin{bmatrix} 1_E & -X^\dagger \\ X & 1_E \end{bmatrix}, \tag{18}$$

in such a way

$$U_X^{-1} R U_X=\begin{bmatrix} A+BX & 0 \\ 0 & C-B^\dagger X^\dagger \end{bmatrix}. \tag{19}$$

From the above consideration one can see that to diagonalize the Hamiltonian (15), one needs to solve the following Riccati equation:

$$\alpha X^2 + X(H_+ +\beta) - (H_- -\beta)X - \alpha = 0. \tag{20}$$

Unfortunately, we do not know how to do that. Note that if $\alpha=0$ then $X=0$ is a solution. This is obvious since in that case $H(\beta)$ is already in the diagonal form. However, even if $\beta=0$ this problem is still very complicated.

IV. DIFFERENTIAL EQUATION APPROACH

Let us now transform the problem of solving a Riccati equation to the problem of solving a Schrödinger equation on $\mathcal{H}\oplus\mathcal{H}$, with the Hamiltonian given by (15). Let $|\Psi_t\rangle=[|\psi_t\rangle,|\phi_t\rangle]^t$, then $|\Psi_t\rangle$ satisfy $i|\dot{\Psi}_t\rangle=H(\beta)|\Psi_t\rangle$. Of course, we can always write $|\Psi_t\rangle=\exp(-iH(\beta)t)|\Psi_0\rangle$, but this form of the solution is useless since U_t does not have a 2×2 BOM form. It may seem that we circled back to the point where we started since in writing the state $|\Psi_t\rangle$ as a column vector we need to diagonalize the matrix $H(\beta)$. Nothing could be further from the truth. To see this let us introduce operators U and J_t as

$$U=\frac{1}{\sqrt{2}}\begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}, \quad J_t=\exp(i\alpha\sigma_3 t). \tag{21}$$

Let us also define $|\tilde{\Psi}_t\rangle=J_tU|\Psi_t\rangle$, and we can easily see that $i|\dot{\tilde{\Psi}}_t\rangle=H_t|\tilde{\Psi}_t\rangle$, where the periodic Hamiltonian is given by

$$H_t = \begin{bmatrix} H_E & z_t^*(V+\beta) \\ z_t(V+\beta) & H_E \end{bmatrix}, \quad z_t = e^{-i2\alpha t}. \tag{22}$$

The Riccati equation associated with H_t reads

$$z_t^* X V_\beta X + X H_E - H_E X - z_t V_\beta = 0, \tag{23}$$

where $V_\beta = V+\beta$. Straightforward calculations show that $X_t = z_t$ is a solution of the Riccati equation (23). According to (19) we have

$$S_t^\dagger H_t S_t = \begin{bmatrix} H_+ + \beta & 0 \\ 0 & H_- - \beta \end{bmatrix}, \tag{24}$$

where $S_t = (1/\sqrt{2}) U_{z_t}$ and U_{z_t} is given by (18), namely,

$$S_t = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 & -z_t^* \\ z_t & 1 \end{bmatrix}. \tag{25}$$

Note that S_t is a unitary 2×2 BOM. We see that if one could solve the Schrödinger equation for $|\tilde{\Psi}_t\rangle$, then our problem would be solved. Formally, we can always do that using chronological operator T, the solution is given by[14]

$$|\tilde{\Psi}_t\rangle = \mathrm{T}\exp\left(-i\int_0^t H_\tau d\tau\right)|\tilde{\Psi}_0\rangle. \tag{26}$$

Unfortunately, the form (26) of the solution has little use due to the presence of the chronological operator T. Nevertheless, it is interesting to note that the connection between the models (15) and (24) is well defined and the solution to Eq. (23) can be easily found; yet finding the solution to Eq. (20) poses a big problem.

V. EXAMPLES

Until now we have not specified the operators H_E and V, which means that the analysis we presented was very general. That fact implies an important concept, namely, that the analysis that we carried out does not depend on the particular choice of a heat bath. It is crucial, however, that the coupling of the qubit with the environment is given by the matrix $f(\sigma_3)$, where f is an arbitrary analytical function. It is interesting to consider the model where operators H_E and $V = V(g)$ are defined as

$$H_E = \int_0^\infty d\omega \omega a^\dagger(\omega) a(\omega), \tag{27}$$

where $a^\dagger(\omega)$ and $a(\omega)$ are bosonic annihilation and creation operators, respectively, and it satisfies the commutation relations $[a(\omega), a^\dagger(\omega')] = \delta(\omega-\omega')$, $\omega, \omega' > 0$. $V(g)$ is given by

$$V(g) = \int_0^\infty d\omega (g^*(\omega) a(\omega) + g(\omega) a^\dagger(\omega)), \tag{28}$$

where $g \in L^2[0,\infty]$. Operators H_E and $V(g)$ given by (27) and (28) define the bosonic heat bath[3] of the qubit. One can find that

$$H_+ = W(g) H_E W(g)^\dagger + C(g), \tag{29}$$

$$H_{-} = W(g)^{\dagger} H_E W(g) + C(g), \tag{30}$$

where $C(g)$ is a constant. One can always rescale the Hamiltonian so that $C(g)=0$; thus we will omit the constant. The unitary Weyl's operator has the form $W(g)=\exp(A(g))$, where

$$A(g) = \int_0^{\infty} d\omega (g^*(\omega) a(\omega) - g(\omega) a^{\dagger}(\omega)). \tag{31}$$

In the case of the $\alpha=0$, the model can be solved exactly.[4] If $\alpha \neq 0$ obtaining the exact reduced dynamics, according to (20) is at least as difficult as solving the equation (to simplify we put $\beta=0$),

$$\alpha X^2 + X(WH_E W^{\dagger}) - (W^{\dagger} H_E W)X - \alpha = 0. \tag{32}$$

The solution of Eq. (32) is yet to be discovered.

As a second example let us consider a pure decoherence case. In this situation $[H_Q \otimes 1_E, H_{\text{int}}]=0$. Let $H_{\text{int}}=M\otimes V$, where M is an arbitrary Hermitian 2×2 matrix. Since operators $H_Q \otimes 1_E$ and $M\otimes V$ commute, we need to diagonalize the following matrix:

$$I_2 \otimes H_E + M \otimes V = \begin{bmatrix} H_E + m_{11}V & m_{12}V \\ m_{12}^* V & H_E + m_{22}V \end{bmatrix}. \tag{33}$$

The Riccati equation associated with BOM (33) is of the form

$$m_{12} XVX + X(H_E + m_{11}V) - (H_E + m_{22}V)X - m_{12}^* V = 0. \tag{34}$$

If $X=x1_E$, where $x\in \mathbb{C}$, one can write the above equation as

$$(m_{12}x^2 + (m_{11} - m_{22})x - m_{12}^*)V = 0, \tag{35}$$

or equivalently as

$$m_{12}x^2 + (m_{11} - m_{22})x - m_{12}^* = 0. \tag{36}$$

As a result, we see that for the dephasing case the Riccati equation simplifies to the quadratic equation and therefore solution X can be easily found.

VI. SUMMARY

In this paper the problem of the exact solution of the decoherence model has been connected with the Riccati operator equation. It was shown that obtaining the exact reduced dynamics is at least as problematic as solving the Riccati equation. Furthermore, we simplified a wide class of problems described by the time dependent Hamiltonian to the time independent problems. One can easily learn from this paper that solving the time dependent Riccati Eq. (23) is very simple, while obtaining the solution of Eq. (20) becomes a very difficult task.

We strongly believe that solving the model we analyzed is crucial and that it can contribute to the progress and verification of the adiabatic theorem for open quantum systems[15] in analogy to the contribution of the half spin particle model with Hamiltonian $H_Q(t,\beta)$ to the progress of the adiabatic theorem for the quantum closed systems.

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