Riccati equation and the problem of decoherence II: Symmetry and the solution of the Riccati equation

Bartłomiej Gardas

View Online   Export Citation

ARTICLES YOU MAY BE INTERESTED IN

Journal of
Mathematical Physics

YOUR RESEARCH Belongs Here

LEARN MORE

Riccati equation and the problem of decoherence II: Symmetry and the solution of the Riccati equation

Bartłomiej Gardas[a)]
Institute of Physics, University of Silesia, PL-40-007 Katowice, Poland

In this paper we revisit the problem of decoherence by applying the block operator matrices analysis. The Riccati algebraic equation associated with the Hamiltonian describing the process of decoherence is studied. We prove that if the environment responsible for decoherence is invariant with respect to the antilinear transformation then the antilinear operator solves the Riccati equation in question. We also argue that this solution leads to neither a linear nor an antilinear operator similarity matrix. Therefore, we cannot use the standard procedure for solving a linear differential equation (e.g., Schrödinger equation). Furthermore, the explicit solution of the Riccati equation is found for the case where the environmental operators commute with each other. We discuss the connection between our results and the standard description of decoherence (one that uses the Kraus representation). We show that the reduced dynamics we obtain does not have the Kraus representation if the initial correlations between the system and its environment are present. However, for any initial state of the system (even when the correlations occur) reduced dynamics can be written in a manageable way. © *2011 American Institute of Physics*. [doi:10.1063/1.3574889]

I. INTRODUCTION

Recently, the connection between a problem of decoherence and the Riccati operator equation was established (for details see Refs. 1 and 2, and references therein). Moreover, it was shown that a wide class of time-dependent quantum systems, precisely the ones that describe a qubit Q interacting with the environment E and defined by the following Hamiltonian:

$$H_{QE}(t, \beta) = H_Q(t, \beta) \otimes \mathbb{I}_E + \mathbb{I}_Q \otimes H_E + f(\sigma_3) \otimes V, \qquad (1)$$

where the Hamiltonian $H_Q(t, \beta)$ of the qubit alone is given by

$$H_Q(t, \beta) = \beta \sigma_3 + \alpha \left(\sigma_1 \cos(\omega t) + \sigma_2 \sin(\omega t) \right), \qquad (2)$$

where $\alpha, \beta \in \mathbb{R}$, can be *effectively* simplified to the time-independent problem. Namely the one governed by the Hamiltonian $H_{QE}(0, \beta) \equiv H_{QE}$. The connection between the reduced dynamics[3] of those models can be summarized in the following formula:

$$\bar{\rho}_t = V_t \rho_t(\bar{\beta}) V_t^\dagger, \qquad (3)$$

where $\bar{\beta} := \beta - \omega/2$ plays the role of effective parameter and $V_t := \mathrm{diag}(e^{-i\omega t/2}, e^{i\omega t/2})$ is the unitary (similarity) transformation. Here, $\bar{\rho}_t$ is the solution (reduced dynamics) of the system identified with the time-dependent Hamiltonian $H_{QE}(t, \beta)$ and $\rho_t(\beta)$ representing the reduced dynamics of the model described by the Hamiltonian H_{QE}. An explicit dependence $\bar{\rho}_t$ of β was omitted. The meaning of the symbols we used in Eqs. (1) and (2) is standard. Furthermore, it was found that this time-independent problem can be solved in two ways. One can either resolve the Riccati operator equation:

$$\alpha X^2 + X(H_+ + \beta) - (H_- - \beta)X - \alpha = 0, \qquad (4)$$

---

a)Electronic mail: bartek.gardas@gmail.com.

    52, 042104-1

where $H_\pm := H_E \pm V$, or solve the following Schrödinger equation (we work with the units $\hbar = 1$):

$$i\dot\Psi_t = H_t\Psi_t, \quad H_t = \begin{bmatrix} H_E & z_t^* V_\beta \\ z_t V_\beta & H_E \end{bmatrix}, \tag{5}$$

with the initial condition $\Psi_0 \equiv \Psi$. In Eq. (5) $\Psi_t = [\psi_t, \phi_t]^T \in \mathcal{H} \oplus \mathcal{H}$, $z_t = e^{-i2\alpha t}$ and $V_\beta = V + \beta \mathbb{I}_E$. In the description above it was assumed that $\mathcal{H}$ is a separable Hilbert space (possibly infinite-dimensional) and H_E and V are the Hermitian operators acting on it. In this paper the Riccati equation $R_H[X] = 0$, where

$$R_H[X] = XBX + XA - CX - B^\dagger \tag{6}$$

is associated with the following Hamiltonian:

$$H = \begin{bmatrix} A & B \\ B^\dagger & C \end{bmatrix}. \tag{7}$$

In turn if the solution of the equation $R_H[X] = 0$ exists, it may be used to diagonalize operator matrix H. The following equality holds true:

$$S_X^{-1} H S_X = \begin{bmatrix} A + BX & 0 \\ 0 & A - (XB)^\dagger \end{bmatrix}, \tag{8}$$

where S_X^{-1} stands for the inversed operator matrix to

$$S_X = \begin{bmatrix} \mathbb{I}_E & -X^\dagger \\ X & \mathbb{I}_E \end{bmatrix}. \tag{9}$$

The Riccati equation (4) is associated with the Hamiltonian H_{EQ}. On the other hand the Riccati equation related to the Hamiltonian H_t reads

$$X(z_t^* V_\beta)X + XH_E - H_E X - z_t V_\beta = 0. \tag{10}$$

One should keep in mind that a given operator A acting on the space $\mathbb{C}^2 \otimes \mathcal{H}$ may be thought of as a block operator matrix, since the following isomorphism holds $\mathbb{C}^2 \otimes \mathcal{H} = \mathcal{H} \oplus \mathcal{H}$.

It is worth mentioning that in spite of the fact that the connection between operator matrices H_t and H_{QE} is well defined and the solution of Eq. (10) can be easily obtained (indeed, it is given by $X_t = z_t \mathbb{I}_E$) the solution of Eq. (4) is still missing. Please note that although X_t is known it cannot be effectively used to resolve Eq. (5) because of its explicit time dependence. Nevertheless it allows us to diagonalize the Hamiltonian H_t, namely,

$$S_{z_t}^\dagger H_t S_{z_t} = \begin{bmatrix} H_+ + \beta\mathbb{I}_E & 0 \\ 0 & H_- - \beta\mathbb{I}_E \end{bmatrix} \equiv H^d, \tag{11}$$

where the unitary matrix S_{z_t} is defined as

$$S_{z_t} = \frac{1}{\sqrt{2}} \begin{bmatrix} \mathbb{I}_E & -z_t^*\mathbb{I}_E \\ z_t\mathbb{I}_E & \mathbb{I}_E \end{bmatrix} = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & -z_t^* \\ z_t & 1 \end{pmatrix} \otimes \mathbb{I}_E. \tag{12}$$

We want to emphasize that Eq. (10) and the slight modification

$$z_t^* X V_\beta X + XH_E - H_E X - z_t V_\beta = 0, \tag{13}$$

used by the author in a previous manuscript on the subject,[2] are equivalent only if the solution is assumed to be a linear operator. This seems to be justified, especially if one expects X to represent an observable. However, it does not need to be true. Thus in this manuscript we will not restrict the analysis to linear operators only.

The purpose of this manuscript is twofold. First, we show that if the environment is (i.e., the operators H_E and V are) invariant under antilinear transformation (Sec. II), then this antilinear operator is the solution of the Riccati equation (10). We also indicate that the mentioned symmetry may be used to diagonalize the Hamiltonian related with Eq. (4), although it does not solve this equation. Next, in Sec. III we argue that the solution we obtained cannot be applied to solve Eq. (5). The problem occurs because standard methods provided by the theory of the differential equations

demand that the operator is not antilinear. The reduced dynamics of the system under consideration is given in Sec. V.

Second, in Sec. IV we study the case where $[H_E, V] = 0$ and we give the exact solution of the Riccati equation (4) for this situation. This is a direct generalization of the recently found solution for the particular operator defining the spin-environment. Finally, in Sec. VI we compare our method with the standard approach based on the operator sum representation. We also discuss the possibility of obtaining an operator (or Kraus) sum representation in the case where the initial correlations between the system and its environment are present.

II. SYMMETRY AND THE SOLUTION TO THE RICCATI EQUATION

Let τ_1 and τ_2 be an antilinear symmetry for H_E and V, such that $\tau_i^2 = \mathbb{I}_E$, (or $\tau_i = \tau_i^{-1}$) for $i = 1, 2$, respectively. By definition, it means that $[H_E, \tau_1] = 0$ and $[V, \tau_2] = 0$. The symmetry operator is an involution, i.e., $\tau^2 = \mathbb{I}_E$. Therefore for a given operator A the condition $[\tau, A] = 0$ is equivalent to the equality $\tau A \tau^{-1} = A$. Operators that fulfill the last equality are invariant under the action of τ and are called τ−symmetric. Furthermore, the statement that an operator τ acting on $\mathcal{H}$ is antilinear has the following meaning:

$$\tau(a\psi + b\phi) = a^*\tau\psi + b^*\tau\phi, \tag{14}$$

for every $\psi, \phi \in \mathcal{H}$ and $a, b \in \mathbb{C}$. We wish to emphasize that in the finite-dimensional case the existence of the aforementioned symmetry is ensured by the theorem of Ali Mostafazadeh[4] which states that every diagonalizable pseudo-Hermitian (in particular the Hermitian) operator H with the discrete spectrum has an antilinear and anti-Hermitian symmetry τ. Moreover, this symmetry is an involution, i.e., $\tau^2 = \mathbb{I}_E$, if H is a Hermitian operator, i.e., $H = H^\dagger$. The proof of this theorem provides the explicit construction of τ. Let us additionally assume that $\tau_1 = \tau_2 := \tau$, which means that the operators H_E and V possess a *common* symmetry τ.

If one allows X to be an antilinear operator then Eq. (10) can be rewritten in the following form:

$$R_{H_t}[X] = z_t \left(X V_\beta X - V_\beta \right) + [X, H_E]. \tag{15}$$

From this equation we can readily see that $R[\tau] = 0$, i.e., the symmetry τ is the solution we were seeking. In particular, if the Hamiltonians H_E and V are (or roughly speaking the E system is) PT−symmetric (or T-symmetric) where P and T stand for the parity and the time-reversal operators, respectively, then $X = PT$ ($X = T$), i.e., the PT (T) operator solves Eq. (10). This is a very interesting and rather unexpected result that the famous PT (T) symmetry is the solution of the Riccati equation we study.

For instance, it can be easily proven that if both H_E and V are symmetric operators, that is to say $H_E^T = H_E$ and $V^T = V$, where by "T" we denote the operation of transposition, then it implies that $R_{H_t}[K] = 0$, where K is a complex conjugate operator:

$$K\psi = \psi^*, \quad \psi \in \mathcal{H}. \tag{16}$$

Above "$*$" stands for the standard complex conjugation operation. This operator possesses the following properties:

(a) $K = K^\dagger$ (i.e., it is a Hermitian operator),
(b) $KK^\dagger = \mathbb{I}_E$ (i.e., it is a unitary operator),
(c) $K^2 = \mathbb{I}_E$ (i.e., it is an involution).

Moreover, K is an antilinear operator. The listed properties follow immediately from (16). Note that for any Hermitian operator (matrix) A the condition $A^T = A$ means that A is K−symmetric, i.e., $[A, K] = 0$. To see this observe that K transforms any operator A in accordance with the following rule:

$$KAK^\dagger = A^*. \tag{17}$$

In addition, if one assumes that A is Hermitian, i.e., $A = A^\dagger$ and uses properties (a) then from the equation above we have

$$KAK = A^T, \tag{18}$$

where we stated the fact that $A^\dagger = (A^T)^*$. Clearly, for the symmetric matrix $KAK = A$ or $[A, K] = 0$. Another way to see that $R_{H_t}[K] = 0$ is to rewrite $R[K]$ as

$$R_{H_t}[K] = z_t(V_\beta^T - V_\beta) + (H_E^T - H_E)K. \tag{19}$$

Evidently, for the symmetric operators the right side of Eq. (19) vanishes.

At the end of this section we show how to diagonalize the operator H_{QE} using the symmetry τ. First, note that by introducing the unitary operator U such that

$$U = \frac{1}{\sqrt{2}}\begin{bmatrix} \mathbb{I}_E & i\mathbb{I}_E \\ i\mathbb{I}_E & \mathbb{I}_E \end{bmatrix} = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 & i \\ i & 1 \end{pmatrix} \otimes \mathbb{I}_E, \tag{20}$$

we obtain $U^\dagger H_{QE} U = \bar{H}$, where

$$\bar{H} = \begin{bmatrix} H_R & V_{\alpha\beta} \\ V_{\alpha\beta}^\dagger & H_R \end{bmatrix} \tag{21}$$

and $V_{\alpha\beta} := \alpha\mathbb{I}_E - iV_\beta$. Because $\tau V_{\alpha\beta}\tau = V_{\alpha\beta}^\dagger$, from Eq. (21) we see that $R_{\bar{H}}[\tau] = 0$, i.e., the symmetry τ is the solution of the Riccati equation associated with the Hamiltonian (21). Therefore, the matrix $\bar{S}_\tau := U S_\tau$ diagonalizes the Hamiltonian H_{QE}. Specifically, the following equation holds true:

$$\bar{S}_\tau^\dagger H_{QE} \bar{S}_\tau = \begin{bmatrix} H_R + V_{\alpha\beta}\tau & 0_E \\ 0_E & H_R - V_{\alpha\beta}^\dagger\tau \end{bmatrix}. \tag{22}$$

Keep in mind that the operator $V_{\alpha\beta}$ is *not* Hermitian, while the operator $V_{\alpha\beta}\tau$ is. Interestingly, we diagonalized the block operator matrix H_{QE} without directly resolving the Riccati algebraic equation associated with it. Note, this is found to be possible if there exists a matrix T such that it transforms a given operator matrix A to another $\bar{A}$ viz $\bar{A} = T^{-1}AT$ and the Riccati equation $R_{\bar{A}}[X] = 0$ related to $\bar{A}$ has the solution $X_{\bar{A}}$. If that is the case, one may construct similarity operator matrix, namely, $S_{X_{\bar{A}}}T$ that diagonalizes A. We see that if $A = H_{QE}$, then $T = U$ [Eq. (20)] and $\bar{A} = \bar{H}$ [Eq. (21)], thus the procedure is working. Therefore, although this strategy is worthy of further study we will not focus on this subject in this paper.

III. THE PROBLEM WITH THE SCHRÖDINGER EQUATION

The results of the paper[2] and Sec. II show that $R_{H_t}[z_t] = 0$ and $R_{H_t}[\tau] = 0$, where $R_{H_t}[X]$ stands for the left side of Eq. (10). Note, the second solution ($X = \tau$) we obtained, in contrast to the first one ($X_t = z_t$), is time independent. Therefore, one may think that the second solution has the advantage because it allows one to construct the operator matrix S_τ that diagonalizes the Hamiltonian (5) and does not depend on time. Indeed, this is the case; however, one serious drawback arises. To see this clearly, we give an explicit form of the matrix S_τ. In agreement with Eq. (9) it takes the form [compare this with Eq. (12)]

$$S_\tau = \frac{1}{\sqrt{2}}\begin{bmatrix} \mathbb{I}_E & -\tau \\ \tau & \mathbb{I}_E \end{bmatrix}. \tag{23}$$

Because the symmetry τ is an involution operator ($\tau^2 = \mathbb{I}_E$), S_τ is the unitary matrix ($S_\tau S_\tau^\dagger = \mathbb{I}_E$). We see that the similarity transformation (23) is neither a linear nor an antilinear operator. In particular, one may easily verify that $S_\tau i = i S_\tau^\dagger$. All these difficulties are a direct consequence of the fact that the operator τ is antilinear. As a result, we cannot use the standard procedure that allows us to solve a linear, differential equation to resolve the Schrödinger equation $i|\dot{\Psi}_t\rangle = H_t|\Psi_t\rangle$. Indeed, because of the presence of the factor i on the left side of the Schrödinger equation we cannot apply the $|\Phi_t\rangle = S_\tau|\Psi_t\rangle$ transformation to reduce it to the form $i|\dot{\Phi}_t\rangle = H_t^d|\Phi_t\rangle$, where H_t^d stands for

the diagonal form of H_t. Another technique is needed to resolve these difficulties. Unfortunately, at the present time there is none. Of course, the same problems occur when one tries to resolve the equation $i|\dot{\Psi}_t\rangle = H_{QE}|\Psi_t\rangle$ using Eq. (22).

Notice that according to the procedure explained in Ref. 2 we have [compare with Eq. (11)]

$$S_\tau^\dagger H_t S_\tau = \begin{bmatrix} H_E + z_t^* V_\beta \tau & 0 \\ 0 & H_E - z_t V_\beta \tau \end{bmatrix} \equiv H_t^d. \qquad (24)$$

Comparing Eqs. (11) and (24) one can learn that the solution $X_t = z_t$ leads to the time-independent diagonal form H^d of the operator H_t, but the transformation matrix S_t does depend on time. An opposite situation takes place in the case of the second solution $X = \tau$, i.e., the diagonal form H_t^d is time-dependent and transformation matrix S_τ does not depend on time.

IV. SOLUTION OF THE RICCATI EQUATION

Recently, the solution of the model specified by Eq. (1) was given in the case where $\mathcal{H} = \bigotimes\limits_{n=1}^{N} \mathbb{C}^2$ and the operator H_E, V are defined as (for detailed discussion see Ref. 5)

$$H_E = \sum_{n=1}^{N} \omega_n \sigma_3(n), \quad V = \sum_{n=1}^{N} g_n \sigma_3(n), \qquad (25)$$

where ω_n and g_n are certain constants defining the frequencies (of the spin-bath) and the coupling constant, respectively. For every $n \leq N$ the operator $\sigma_3(n)$ is understood as $\sigma_3(n) = I_2 \otimes ... \otimes \sigma_3 \otimes ... \otimes I_2$, where I_2 is a 2×2 identity matrix and σ_3 is the usual Pauli matrix. Note that the operators above commute, i.e., $[H_E, V] = 0$. We will extend results obtained in Ref. 5 to the arbitrary operators that commute with each other.

If one assumes that the operators H_E and V commute, then they have a common set of eigenvectors. Henceforward, we will assume that the eigenvalues of H_E and V are all discrete and not degenerated. Furthermore, the spectrum of a given operator A will be denoted by $\sigma(A)$. As a summary, one can write the following eigenvalue problems for H_E and V:

$$H_E|\phi_n\rangle = E_n|\phi_n\rangle, \quad V|\phi_n\rangle = V_n|\phi_n\rangle, \qquad (26)$$

where $E_n \in \sigma(H_E)$ and $V_n \in \sigma(V)$. Here, the index n goes through the set of all integer numbers or through some subset of it. We wish to emphasize that the assumption of the discreteness and lack of degeneration of the spectrum of the operators in question is not crucial in our analysis. In fact, it can be easily overcome (however, we will not address this technical issue in the current manuscript).

Note, the solution X of Eq. (4) is a function of the operators $H_\pm$ defined in Eq. (4). Since the operators H_E and V commute with each other, so do the operators $H_\pm$. Therefore, $[X, H_\pm] = 0$ and the Riccati equation (4) can be simplified to the following form:

$$\alpha X^2 + 2V_\beta X - \alpha = 0. \qquad (27)$$

One can observe that if $\alpha = 0$ then $X_0 = 0_E$ is the solution of Eq. (27). In this case matrix H_{QE} is already in the diagonal form. Yet, the operator X_0 may not be the only solution of the equation $R_{H_{QE}}[X] = 0$.

In order to obtain the solution for the case where $\alpha \neq 0$ we will apply the spectral theorem for Hermitian operators. From Eq. (27) one can readily see that $X = f(V_\beta)$, where the function f takes the form

$$f(\lambda) = \frac{-\lambda + \sqrt{\lambda^2 + \alpha^2}}{\alpha}, \quad \lambda \in \sigma(V_\beta), \qquad (28)$$

and $\sigma(V_\beta) = \sigma(V + \beta \mathbb{I}_E)$. One may also write the operator $f(V)$ as:

$$f(V) = \sum_{\lambda \in \sigma(V_\beta)} f(\lambda)|\lambda\rangle\langle\lambda|. \qquad (29)$$

Thus, in view of Eqs. (26) and (29) X takes the final form

$$X = \sum_n x_n |\phi_n\rangle\langle\phi_n|, \tag{30}$$

where abbreviation $x_n \equiv f(V_n + \beta)$ was introduced. Note that x_n are the eigenvalues of the operator X, that is, $X|\phi_n\rangle = x_n|\phi_n\rangle$. The operator above is Hermitian, i.e., $X = X^\dagger$. Since for every parameter α and β the function f specified by Eq. (28) takes positive values, the eigenvalues x_n are positive. Therefore, the solution we obtained in Eq. (30) is a positively defined operator. Moreover, as was pointed out in Ref. 5 there exists at least one more solution of Eq. (27), where H_E and V were chosen to be the ones that describe the spin-bath [see Eq. (25)]. This situation also occurs when the operators H_E and V have a more general form. The second solution is given by

$$\bar{f}(\lambda) = \frac{-\lambda - \sqrt{\lambda^2 + \alpha^2}}{\alpha}, \quad \lambda \in \sigma(V_\beta). \tag{31}$$

It can be easily verified that this function determines the negatively defined operator, that is to say

$$\bar{X} = \sum_n \bar{x}_n |\phi_n\rangle\langle\phi_n|, \tag{32}$$

and $\bar{x}_n := \bar{f}(V_n + \beta)$. Observe that $\bar{X} = \bar{X}^\dagger$, i.e., this operator is also Hermitian. Note also that $f(\lambda)\bar{f}(\lambda) = -1$, for $\lambda \in \sigma(V_\beta)$. This may be verified directly, or may also be thought of as a consequence of the Vieta's formulas, if one thinks of f and $\bar{f}$ as the solution of the quadratic equation: $\alpha x^2 + \lambda x - \alpha = 0$, for $\lambda \in \sigma(V_\beta)$. We want to emphasize, however, that the Riccati equation (27) is *not* a simple quadratic polynomial, even though it may appear so. As a result there might exist other solutions of this equation that are not specified by the well-known formula for roots of the quadratic equation.

At this point natural questions may by asked, for instance, which operator, X or $\bar{X}$ (or any other, if it exists) should be used to diagonalize the block operator matrix H_{QE}? What difference (if any) does it make? Of course, each solution of the Riccati equation will diagonalize the operator matrix with which it is associated. Nevertheless, in certain cases it may by convenient to choose one solution instead of the other. For example, by studying limiting cases like $\alpha \to 0$. Indeed, if $\beta \neq 0$, then we find that (recall that $f \sim 1/\bar{f}$)

$$\lim_{\alpha \to 0} f = 0, \quad \lim_{\alpha \to 0} \bar{f} = -\infty. \tag{33}$$

Therefore, $X \to 0_E$ while $\bar{X} \to -\infty$ as α goes to 0. This means that the first solution is a continuous function of the parameter α, including the $\alpha = 0$ value even though the second operator, i.e., $\bar{X}$, does *not* exist in that point at all. As a result, if in the process of analysis a person decides to use the second solution $\bar{X}$ it may encounter serious problems taking the limits $\alpha \to 0$. Furthermore, as was mentioned earlier, the first solution is a positively defined operator and is more suitable to manage. Henceforward, we will restrict further analysis to the operator X given by Eq. (30) only.

V. THE EXACT REDUCED DYNAMICS

We now use the solution above to construct the exact reduced dynamics of the model described by H_{QE}. Obtaining the exact reduced dynamics of the model under consideration, namely, the one defined by Eq. (1), can be easily accomplished using Eq. (3), as we mentioned earlier. We begin with the construction of the evolution operator U_t generated by the Hamiltonian H_{QE}. Foremost, let us recall that the evolution operator U_t may be computed by applying the following formula:

$$U_t = S_X \exp(-i H_d t) S_X^{-1}, \tag{34}$$

where H_d stands for the diagonal form of the Hamiltonian H_{QE} and in agreement with Eq. (8) it takes the form

$$H_d = \begin{bmatrix} H_+ + \alpha X & 0_E \\ 0_E & H_- - \alpha X \end{bmatrix}, \tag{35}$$

where X is given by (30). Next let us observe that the inverse operator S_X^{-1} is

$$S_X^{-1} = \begin{bmatrix} g(X) & 0_E \\ 0_E & g(X) \end{bmatrix} S_X^\dagger \tag{36}$$

$$\equiv G(X)S_X^\dagger,$$

where $g(X)$ is the function of X and it is given by

$$g(\lambda) = \frac{1}{1+\lambda^2}, \quad \lambda \in \sigma(X). \tag{37}$$

Note that $[X, V] = 0$, thus X and V have the same eigenstate, namely, $|\phi_n\rangle$. Since $X = f(V_\beta)$, the eigenvalues x_n of X are given by $x_n = f(V_n + \beta)$. Obviously, $\sigma(X) = f(\sigma(V_\beta))$. The same arguments lead to the conclusion that $g(X)|\phi_n\rangle = (g \circ f)(E_n + \beta)|\phi_n\rangle$, this implies that $\sigma(g(X)) = (g \circ f)(\sigma(V_\beta))$. Using Eqs. (34)–(36) the final form of the evolution operator of the total system can be written as

$$U_t = G(X) \begin{bmatrix} U_t^+ + X^2 U_t^- & (U_t^+ - U_t^-)X \\ (U_t^+ - U_t^-)X & U_t^- + X^2 U_t^+ \end{bmatrix}, \tag{38}$$

where $U_t^\pm := \exp(-i(H_\pm \pm \alpha X)t)$. Equation (38) represents the evolution operator of the total system $Q + E$ and it can be easily applied to any (not only to the factorable one) initial state ρ_{QE} of that system, since it is written in a 2×2 block operator matrix. At this point reduced dynamics $\rho_Q(t)$ may be obtained in the form

$$\rho_Q(t) = \text{Tr}_E(U_t \rho_{QE} U_t^\dagger), \tag{39}$$

where Tr_E is the partial trace operation. Note that in general case this may not be easy to accomplish, even though Eq. (39) might indicate so. The reason is that to compute partial trace Tr_E one needs to write operator $\rho_{QE}(t)$ in the 2×2 block operator form.

VI. KRAUS REPRESENTATION

It is interesting to see how the results of Sec. IV and Sec. V are related to the standard description of the completely positive map via so called Kraus representation:

$$\rho_Q(t) = \sum_\mu K_\mu(t)\rho_Q K_\mu(t)^\dagger. \tag{40}$$

The Kraus matrices $K_\mu(t)$ satisfy the following completeness relation $\sum_\mu K_\mu(t)K_\mu(t)^\dagger = \mathbb{I}_Q$. It is well established that it is possible to derive Eq. (40) from Eq. (39) assuming that no correlations between the system and its environment are present initially.[3,6] The generalization to the case when the initial state is not factorable is also possible.[7,8] Nevertheless, in practice finding the Kraus matrices is impossible in most cases. We will show how to construct those matrices for the system we study. To accomplish this goal let us rewrite S_X in the following manner:

$$S_X = \sum_n \begin{pmatrix} 1 & -x_n \\ x_n & 1 \end{pmatrix} \otimes |\phi_n\rangle\langle\phi_n|$$

$$\equiv \sum_n F_n \otimes |\phi_n\rangle\langle\phi_n|. \tag{41}$$

We also used resolution of the identity $\mathbb{I}_E$ in the $|\phi_n\rangle$ basis, that is to say $\mathbb{I}_E = \sum_n |\phi_n\rangle\langle\phi_n|$. Note the inverse operator S_X^{-1} can be written in a similar fashion, namely,

$$S_X^{-1} = \sum_n F_n^{-1} \otimes |\phi_n\rangle\langle\phi_n|, \tag{42}$$

where F_n^{-1} is the inverse matrix of F_n. Since we have $\det(F_n) = 1 + x_n^2 > 0$ it always exists. Due to the fact that $F_n^{-1} = F_n^\dagger/\det(F_n)$ holds we can rescale F_n, namely, $F_n \rightarrow 1/\sqrt{\det(F_n)}F_n$ so it

becomes the unitary operator. Furthermore, the Hamiltonian (35) may be rewritten as

$$H_d = \sum_n \begin{pmatrix} h_n^+ & 0 \\ 0 & h_n^- \end{pmatrix} \otimes |\phi_n\rangle\langle\phi_n|$$

$$\equiv \sum_n H_n^d \otimes |\phi_n\rangle\langle\phi_n|,$$

(43)

where $h_n^\pm := (E_n^\pm \pm \beta) \pm \alpha x_n$ and $E_n^\pm := E_n \pm V_n$ are the eigenvalues of $H_\pm$ (note $\sigma(H_\pm) = \sigma(H_E \pm V)$)). Combining Eq. (34) and Eqs. (41)–(43) we obtain

$$U_t = \sum_n U_n(t) \otimes |\phi_n\rangle\langle\phi_n|,$$

(44)

with the unitary matrices $U_n(t) = \exp(-i H_n t)$ and $H_n = F_n^{-1} H_n^d F_n$. One can see from the form of the evolution operator above and Eq. (39) that we finally have

$$\rho_Q(t) = \sum_n \rho_n U_n(t) \rho_Q U_n(t)^\dagger$$

$$\equiv \sum_n K_n(t) \rho_Q K_n(t)^\dagger.$$

(45)

In Eq. (45) $\rho_n := \langle\phi_n|\rho_E|\phi_n\rangle$ and the Kraus matrices are defined as $K_n(t) := \sqrt{\rho_n} U_n(t)$. Therefore, the operator sum representation of the model we study is found.

A. Connection with the Riccati diagonalization

Interestingly, the explicit form of the matrix H_n reads

$$H_n = \begin{pmatrix} E_n^+ + \beta & \alpha \\ \alpha & E_n^- - \beta \end{pmatrix},$$

(46)

and it does *not* depend on the eigenvalues x_n of the operator X. Therefore, one may draw the conclusion that the Kraus matrices obtained in Eq. (45) do not depend on x_n. To solve this puzzle, notice first that in order to compute the Kraus matrices one needs to determine the "evolution" operator $U_n(t)$. The latter requires diagonalization of its "generator" H_n. Because $H_n = F_n^{-1} H_n^d F_n$ the dependence of the Kraus matrices on the eigenvalues x_n is somewhat "hidden," in the similarity matrix F_n. It is important to realize that the diagonalization procedure $H_n \xrightarrow{F_n} H_n^d$ differs from the standard diagonalization routine, which is based on the eigenvalue problem for the operator H_n. This new kind of algorithm is called *Riccati diagonalization* and was recently introduced in Ref. 9. One may see that it arises naturally in our analysis. Let us also recall that the similarity matrix F_n is composed of the nth eigenvalue x_n of the operator X, which is the solution of Riccati equation (27), while the eigenvalues of the matrix H_n are the solution of the following characteristic equation associated with H_n:

$$\lambda^2 - \lambda \mathrm{Tr}(H_n) + \det(H_n) = 0.$$

(47)

We have already found its solution indirectly in Eq. (43). The roots of this equation are given by $\lambda_n^\pm = h_n^\pm$. The corresponding eigenvector (not normalized) reads

$$\boldsymbol{\lambda}_n^+ = \begin{pmatrix} -\bar{x}_n \\ 1 \end{pmatrix}, \quad \boldsymbol{\lambda}_n^- = \begin{pmatrix} -x_n \\ 1 \end{pmatrix}.$$

(48)

Therefore, the similarity (not unitary) matrix $G_n = (\boldsymbol{\lambda}_n^+, \boldsymbol{\lambda}_n^-)$, the one that also diagonalizes H_n takes the form

$$G_n = \begin{pmatrix} -\bar{x}_n & -x_n \\ 1 & 1 \end{pmatrix}.$$

(49)

The connection between similarity matrices F_n and G_n follows. If $F_n \equiv (\boldsymbol{\xi}_n^1, \boldsymbol{\xi}_n^2)$, then $\boldsymbol{\xi}_n^1 = x_n \cdot \boldsymbol{\lambda}_n^+$ and $\boldsymbol{\xi}_n^2 = \boldsymbol{\lambda}_n^-$, since $x_n \bar{x}_n = -1$. Therefore the matrix F_n is also composed of the eigenvectors of H_n,

yet the matrices G_n and F_n are *not* similar, i.e., the invertible matrix P_n, such that $G_n = P_n F_n P_n^{-1}$, does not exist. It immediately follows from the fact that $\mathrm{Tr}(G_n) \neq \mathrm{Tr}(F_n)$ as well as $\det(G_n) \neq \det(F_n)$. Because the similarity transformation preserves the trace and determinant, the matrices G_n and F_n cannot be similar. From the consideration above the main difference between the two methods can be easily seen. In the standard approach one needs to determine the eigenvalues $\lambda_n^{\pm}$ of H_n, as was mentioned above [i.e., one needs to solve Eq. (47)]. On the other hand, to use the Riccati diagonalization to our advantage we need to find x_n [i.e., resolve Riccati equation (27)]. Note that in the case of the ordinary matrix, Eq. (27) becomes the quadratic equation, but it differs from the characteristic Eq. (47). In our model the relation between $\lambda_n^{\pm}$ and x_n may be summarized as

$$\lambda_n^{\pm} = (E_n^{\pm} \pm \beta) \pm \alpha x_n. \tag{50}$$

B. Initial correlation

If the initial correlations between a system of interest and its environment are present, i.e., ρ_{QE} takes the form

$$\rho_{QE} = \sum_{ij} \gamma_{ij} \rho_Q^i \otimes \rho_E^j, \tag{51}$$

for some *not* factorable complex number γ_{ij}, namely, $\gamma_{ij} \neq \gamma_1^i \gamma_2^j$, then reduced dynamics (39) *cannot* be written in the form (40). This not so obvious, since there are cases in which, even though initial correlations are present, the reduced dynamics can still be written in the Kraus form.[10]

Yet, for a finite dimensional environment there exists a simple criterion.[8] It allows one to verify when the given state $\rho_Q(t)$ possesses the operator sum representation (40) if initial correlations are present. The necessary and sufficient condition for the later to hold true for *any* initially correlated state is that the joint dynamics has to be locally unitary, i.e., the evolution operator U_t for the total system needs to be of the form

$$U_t = U_Q(t) \otimes U_E(t). \tag{52}$$

The operator $U_Q(t)$ ($U_E(t)$) describes the evolution of the system Q (E) alone. Observe that if the evolution of the total system is *not* locally unitary that this does not necessarily imply that $\rho_Q(t)$ does not possess Kraus representation for a *particular* initially correlated state ρ_Q. From Eq. (44) one can readily see that the evolution operator (38) does not have the form (52) as may be expected. Note that from (51) and Eq. (44) we obtain

$$\rho_Q(t) = \sum_n \sum_{ij} \varepsilon_{ij}^n U_n(t) \rho_Q^i U_n(t)^{\dagger}, \tag{53}$$

where $\varepsilon_{ij}^n = \gamma_{ij} \langle \phi_n | \rho_E^j | \phi_n \rangle$. Therefore, even if in this general case the reduced dynamics cannot be written in the operator sum representation, one can still describe the evolution of the system in a manageable way.

VII. SUMMARY

In this paper we investigated the Riccati algebraic equation associated with the Hamiltonian modeling the process of decoherence in the case of one qubit. It was shown that if the environment is τ-symmetric, where τ is an antilinear involution, then τ is the solution of the Riccati equation under consideration. We indicated that even though the solution of the Riccati equation has been found it cannot be applied to obtain the reduced dynamics, due to the problem with a standard procedure allowing one to solve the Schrödinger equation. We wish to emphasize that this result does *not* contradict with the previous paper,[2] where we claim that the solution of the Riccati equation enables us to rewrite the evolution operator generated by the Hamiltonian H_{QE} as a 2×2 block operator matrix. Of course, the reason that problems occur is that the solution of the Riccati equation is antilinear.

Furthermore, we provided a full resolution of the problems introduced in Ref. 2 for the case when operators defining the environment commute with each other (regardless of the existence of any symmetry in the system). We also derived the operator sum representation for that model assuming no correlations between the systems are present initially. Moreover, we also showed how to obtain the solution if the initial state of the total system is not factorable. This result is a direct generalization of the system discussed in Ref. 5. We also indicated that the recently derived schema of so called Riccati diagonalization arises naturally in the model we considered.

ACKNOWLEDGMENTS

[1] B. Gardas, Master's thesis, Institute of Physics University of Silesia, Poland, 2009.
[2] B. Gardas, J. Math. Phys. **51**, 62103 (2010).
[3] R. Alicki and K. Lendi, *Quantum Dynamical Semigroups and Applications*, 2nd ed., Springer Lecture Notes in Physics Vol. 717 (Springer, Berlin, 2007).
[4] A. Mostafazadeh, J. Math. Phys. **43**, 3944 (2002).
[5] B. Gardas, Phys. Rev. A **82**, 042115 (2010).
[6] I. Bengtsson and K. Życzkowski, *Geometry of Quantum States: An Introduction to Quantum Entanglement* (Cambridge University Press, Cambridge, 2006).
[7] P. Štelmachovič and V. Bužek, Phys. Rev. A **64**, 062106 (2001); **67**, 029902 (2003).
[8] H. Hayashi, G. Kimura, and Y. Ota, Phys. Rev. A **67**, 062109 (2003).
[9] K. Fujii and H. Oike, "Riccati diagonalization of Hermitian matrices," Int. J. Geom. Methods Mod. Phys., arXiv:1004.1207v2.
[10] D. M. Tong, J. L. Chen, J. Y. Huang, L. C. Kwek and C. H. Oh, Laser Phys. **16**, 1512 (2006).