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Performance and Cost Evaluation of an Adaptive Queuing System with Customer Reneging and Retention: Steady-state and Transient Analysis

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Abstract Customers are often required to wait when they arrive at service facilities and see the servers are busy, or when they find other customers who arrived earlier waiting. The longer customers wait, the more dissatisfied they are likely to be and may leave the queue without receiving service (reneging). The objective of the service provider is to improve the quality of service in order to minimize the possibility of customer reneging since it increases cost and reduces revenue. Therefore, a trade-off between performance and cost should be considered when designing, planning and reducing the queues at service facilities. In this paper, we propose an adaptive queuing model with the retention of reneging customers. We derive the steady-state and transient-state performance parameters, and also discusses performance and cost evaluation. We demonstrate the utility of the model in the evaluation of waiting lines in the service industry using numerical examples.

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1 Introduction

Queuing is a common phenomenon in the service industry. Customers usually experience queuing instances in which they have to wait at the service facility for a random amount of time before being serviced [3,21]. The customers are sensitive to waiting time, and if it is longer than the customer expected, the customer may leave the queue without receiving service. When a customer leaves the queue without receiving service, it is called reneging. The service provider could reduce the queue size by deploying extra resources to increase the service speed when the queue size becomes significant. A queuing system in which the service rate depends on the queue size is called an adaptive queuing system. The service provider may provide some incentives to prevent the customers from leaving the queue, such as entertainment, bonuses, and a comfortable environment or the service provider may increase the service speed. When the service provider tries to prevent the customers from leaving the service facility before their service begins is called customer retention. In industry, if the service is good, customer has to revisit again. Li [15] discussed about customers' satisfaction. He observed that customer satisfaction displays significantly positive relations to customer revisit intention.

Queuing systems with reneging are those in which customers can abandon the system before their service begins. These kinds of models exist in a wide variety of domains, such as call centres, communication networks, health care, inventory systems [19], restaurants, supermarkets and many other service industries. A customer waits in the queue, and after a random amount of time, the customer may leave the system without entering service. It is important to note that the decision to leave the queue before entering service may not necessarily result from longer waiting times but also from poor quality of service as perceived by the customer and other factors that constraints the customer. If certain customer retention strategies are employed, then there are chances that a reneging customer can be retained. The authors in [8,9,13,14,16,17] obtained steady-state solutions of queuing model with customer retention and reneging while the authors in [4,10,11,20] discussed the transient-state solutions.

Another way to retain customers that are impatient due to longer queue size or waiting time is to reduce the arrival rate of customers at the service facility or to increase the service rate by deploying more resources. For queuing systems in the service industry where profit is the primary incentive, the second option is preferable, but a tradeoff must be made between the waiting time and the cost of adding additional resources. Service improvement can be modelled by an adaptive queuing systems in which the service time is not fixed but should be increased or decreased depending on the size or waiting time. The authors in [1, 2, 6, 7] discussed steady-state solution of a queuing system with adaptive service rates in which the service rate changes when the number of customers in the queue reaches defined thresholds. They derived steady-state performance evaluation parameters such as the number of customers in the system and the waiting time of customers. However, transient-state analysis of adaptive queuing systems with reneging and retention of reneging customer has not yet been studied in the literature, and it is the subject of the paper. The transient-state analysis is essential in understanding the behaviour of the performance parameters within a short observation time. Tweneboah-Koduah et al. [18] also discussed about the retention strategy in his paper. He observed that due to the intensified competition in the financial sector as a result of homogenous products and services, it is becoming increasingly important for service providers to take steps to retain their customers. So, he investigated the factors influencing NBFIs' customer switching behaviour and found that excessive pricing, poor service quality, customer dissatisfaction and lack of trust to have a statistically significant influence on NBFIs' customer switching behaviour in Ghana.

In this paper we derived both steady-state and transient-state solution of an adaptive queuing system with customer reneging and retention which are very applicable in the service industry.

2 QUEUING MODEL DESCRIPTION

We consider a single-server queuing system with a finite capacity of N. This type of queuing system is represented in Kendall's notation as $M/M/1/N$, where M indicates that the inter-arrival times follows a Poisson process and the service times are exponentially distributed. The following assumptions are considered for the queuing model:

1. The arrival process follows a Poisson process with a mean rate of λ.
2. After waiting in the queue for a random time T without being served, the customer may become discouraged and leave the queue without entering service with a probability of p or remain in the queue with a probability $1-p$ provided some customer retention strategies are deployed. The reneging times are considered to be exponentially distributed with parameter ξ (occurring rate of the reneging time T) [12].

$$r(t) = \xi e^{-\xi t} \tag{1}$$

The reneging rate is given by

$$\xi_n = \begin{cases} 0, & 0 < n \leq 1 \\ (n-1)\xi, & n \geq 2 \end{cases}$$

3. The customers follows first-come-first-serve (FCFS) service discipline. It is the most common service discipline applied in the service industry, e.g. restaurants, supermarkets, hospitals etc.
4. The capacity of the queue is limited to N. The customers that arrive when the capacity is reached will not be admitted into the queue.
5. The service times depends on the queue size. When the queue size grows to a defined threshold of K, the service time is increased from μ_1 to μ_2 by adding more resources. However, when the queue size drops below K, the service time is decreased from μ_1 to μ_2. Therefore, the mean service rate is
$$\mu_n = \begin{cases} \mu_1,\, 0 < n < K \\ \mu_2,\, K \leq n \leq N \end{cases}$$
where $\mu_1 < \mu_2$.
6. It is considered that initially there is no customer in the system, i.e. $P_0(0) = 1$.

3 MATHEMATICAL FORMULATION OF THE QUEUING MODEL

Suppose that the one dimensional Markovian process $\{X(t) = n, t \geq 0, n \geq 0\}$ represent the number of customers present in the queuing system and that $P_n(t) = P\{X(t) = n\}$ is the probability that there are n customers in the system at time t. From Continuous time Markov chains (CTMC) [5], the Kolmogorov forward differential equations for the time-dependent state probabilities governing the distribution of the number of customers in the queue is as follows:

$$\frac{dP_0(t)}{dt} = -\lambda P_0(t) + \mu_1 P_1(t) \tag{2}$$

$$\begin{aligned} \frac{dP_n(t)}{dt} = &- (\lambda + \mu_1 + (n-1)\xi p)\, P_n(t) + \lambda P_{n-1}(t) + \\ &(\mu_1 + n\xi p) P_{n+1}(t); 1 \leq n \leq K-1 \end{aligned} \tag{3}$$

$$\begin{aligned} \frac{dP_n(t)}{dt} = &- (\lambda + \mu_2 + (n-1)\xi p)\, P_n(t) + \lambda P_{n-1}(t) + \\ &(\mu_2 + n\xi p) P_{n+1}(t); K \leq n \leq N-1 \end{aligned} \tag{4}$$

$$\frac{dP_N(t)}{dt} = \lambda P_{N-1}(t) - (\mu_2 + (N-1)\xi p) P_N(t) \tag{5}$$

where $\mu_1 < \mu_2$.

By solving the differential equations above, we determine the state probabilities for the number of customers in the system, from which we derive the performance parameters. In the next two sections, we will present the steady-state analysis and the transient state analysis of the above differential equations.

4 STEADY-STATE ANALYSIS

In this section, we analyze the steady-state behaviour of the system. The steady-state performance and cost evaluation parameters are derived and the influence of the decision variables on the performance and cost parameters analyzed. In steady-state $\lim_{t\to\infty} P_n(t) = P_n$, and $\lim_{t\to\infty} \frac{dP_n(t)}{dt} = 0$. Therefore, the equations (2)-(5) in steady-state become:

$$0 = -\lambda P_0 + \mu_1 P_1 \tag{6}$$

$$0 = -\left(\lambda + \mu_1 + (n-1)\xi p\right) P_n + \lambda P_{n-1} + (\mu_1 + n\xi p)P_{n+1}; \quad 1 \leq n \leq K-1 \tag{7}$$

$$0 = -\left(\lambda + \mu_2 + (n-1)\xi p\right) P_n + \lambda P_{n-1} + (\mu_2 + n\xi p)P_{n+1}; \quad K \leq n \leq N-1 \tag{8}$$

$$0 = -(\mu_2 + (N-1)\xi p)P_N + \lambda P_{N-1} \tag{9}$$

From equation (6), we have

$$P_1 = \frac{\lambda}{\mu_1} P_0$$

Re-write equation (7) as

$$(\mu_1 + n\xi p)P_{n+1} = \left(\lambda + \mu_1 + (n-1)\xi p\right) P_n - \lambda P_{n-1}; \quad 1 \leq n \leq K-1 \tag{10}$$

Now substitute $n = 1, 2, 3, ...K-1$ in (10), we get

$$P_n = \prod_{r=1}^{n} \left[\frac{\lambda}{(\mu_1 + (r-1)\xi p)}\right] P_0$$

Also, equation (8) can be written as

$$(\mu_2 + n\xi p)P_{n+1} = \left(\lambda + \mu_2 + (n-1)\xi p\right) P_n - \lambda P_{n-1}; \quad K \leq n \leq N-1 \tag{11}$$

Now, on substituting $n = K, K+1, ...N-1$ in equation (11), we get

$$P_n = \prod_{r=1}^{K-1} \left[\frac{\lambda}{(\mu_1 + (r-1)\xi p)} \prod_{l=K}^{n} \left\{\frac{\lambda}{\mu_2 + (l-1)\xi p}\right\}\right] P_0$$

Therefore,

$$P_n = \begin{cases} \prod_{r=1}^{n} \left[\frac{\lambda}{(\mu_1+(r-1)\xi p)}\right] P_0, & 1 \leq n < K \\ \prod_{r=1}^{K-1} \left[\frac{\lambda}{(\mu_1+(r-1)\xi p)} \prod_{l=K}^{n} \left\{\frac{\lambda}{\mu_2+(l-1)\xi p}\right\}\right] P_0, & K \leq n \leq N. \end{cases}$$

Using $\sum_{n=0}^{N} P_n = 1$, the probability P_0 can be obtained as

$$P_0 = \left[1 + \sum_{n=1}^{K-1} \prod_{r=1}^{n} \frac{\lambda}{(\mu_1 + (r-1)\xi p)} + \sum_{n=K}^{N} \prod_{r=1}^{K-1} \frac{\lambda}{(\mu_1 + (r-1)\xi p)} \prod_{l=K}^{n} \left\{ \frac{\lambda}{\mu_2 + (l-1)\xi p} \right\} \right]^{-1} \tag{12}$$

4.1 Performance Evaluation

The quality of service perceived by the customer depends on the time customers spent in the queue waiting for service (waiting time) and on the probability that the maximum capacity of the queue is reached (blocking probability). We consider performance parameters such as the number of customers in the queue, the waiting time, the blocking probability.

The blocking probability or the fraction of customers who are rejected (have not joined the queue) because the maximum queue size have reached is obtained from equation (9) for $n = N$ as

$$P_N = \prod_{r=1}^{K-1} \left[\frac{\lambda}{(\mu_1 + (r-1)\xi p)} \prod_{l=K}^{N} \left\{ \frac{\lambda}{\mu_2 + (l-1)\xi p} \right\} \right] P_0 \tag{13}$$

The expected system size which include the customers waiting in the queue and the customer being served is

$$L_s = \sum_{n=0}^{N} n P_n \tag{14}$$

$$= \sum_{n=1}^{N} n \left[\prod_{r=1}^{n} \left\{ \frac{\lambda}{(\mu_1 + (r-1)\xi p)} \right\} + \prod_{r=1}^{K-1} \left\{ \frac{\lambda}{(\mu_1 + (r-1)\xi p)} \times \prod_{l=K}^{n} \left\{ \frac{\lambda}{\mu_2 + (l-1)\xi p} \right\} \right\} \right] P_0 \tag{15}$$

Since customers that arrive when the maximum capacity of the queue is reached do not actually join the queue, the effective arrival rate λ_{eff} is

$$\lambda_{eff} = \lambda(1 - P_N) \tag{16}$$

The steady-state expected queue size, L_q and the expected waiting time in the queue before entering service W_q are:

$$L_q = \sum_{n=2}^{N}(n-1)P_n \tag{17}$$
$$W_q(t) = \frac{Lq}{\lambda_{eff}}$$

In Figures 1, 2 and 3 the variation in expected queue size, expected waiting time in queue and probability of customer rejection with respect to average arrival rate λ is compared for the three queuing systems. In Figure 1 one can observe that the expected queue size is lowest in case of queuing system with service improvement and retention of impatient customers. It is due to the faster service rate adopted after some customers served. Similarly, in Figure 2 it is again observed that the expected waiting time is lowest in case of queuing system with service improvement and retention of impatient customers. Figure 3 shows the comparison of probability of customer rejection for the three queuing systems. Since in queuing system with service improvement and retention of impatient customers the expected system size is lower, therefore, the probability of customer rejection is also lowest in this case. The values of parameters considered are: $\mu_1 = 14, \mu_2 = 15, \xi = 0.05, p = 0.4, N = 40$ with initial condition $P_0(0) = 1$. For Markovian queuing system with retention of impatient customers the value of $\mu = 14.5$.

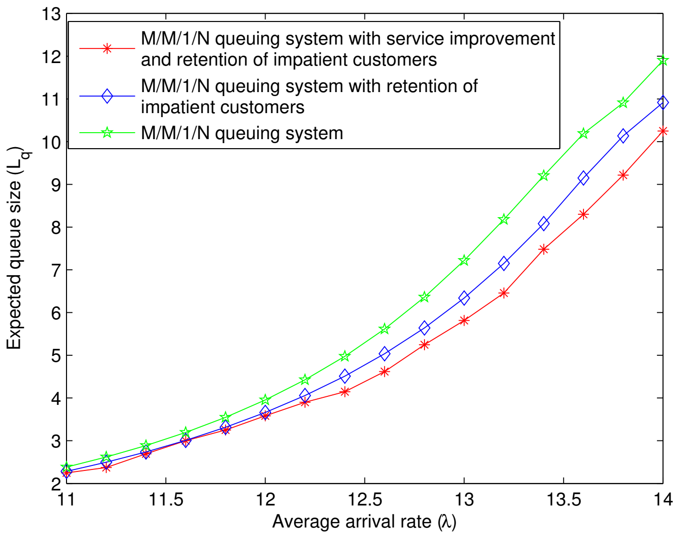

Fig. 1: Variation in expected queue size w.r.t average arrival rate

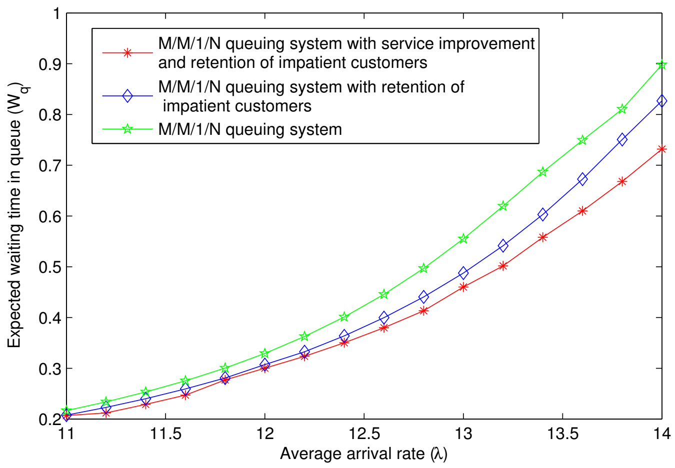

Fig. 2: Variation in waiting time in queue w.r.t average arrival rate

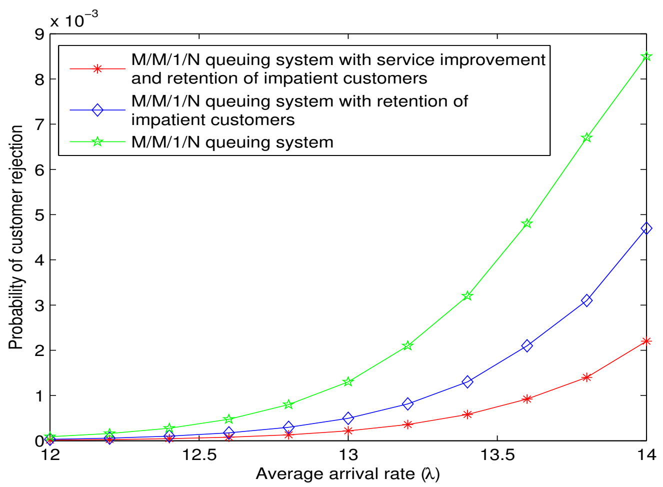

Fig. 3: Variation in probability of customer rejection w.r.t average arrival rate

In figures 4 and 5 we studied the comparison of variation in expected queue size and expected waiting time in queue with respect to probability of retention for the Markovian queuing system with service improvement and retention of impatient customers and Markovian queuing system with retention of impatient customers. It is observed that as the probability of retention of impatient customers increases both the expected queue size and expected waiting time in queue increases. However, one can also observe that expected queue size and expected waiting time in queue is lower for the Markovian queuing system with service improvement and retention of impatient customers. The values of parameters considered are: $\lambda = 13, \mu_1 = 14, \mu_2 = 15, \xi = 0.05, N = 40$ with initial condition $P_0(0) = 1$. For the Markovian queuing system with retention of impatient customers the value of $\mu = 14.5$.

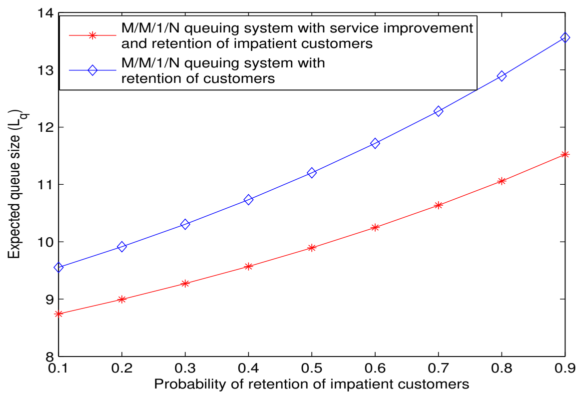

Fig. 4: Variation in expected queue size w.r.t probability of retention of impatient customers

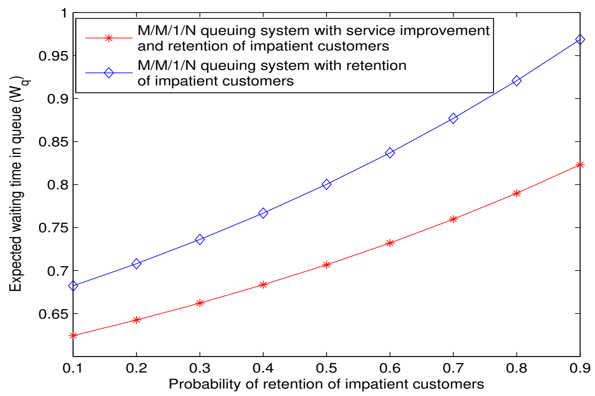

Fig. 5: Variation in expected waiting time in queue w.r.t probability of retention of impatient customers

4.2 Cost Evaluation

When the maximum queue size is reached, and customers can no longer be admitted into the queue at the service facility, or when a customer leaves the queue without receiving service, the service provider experiences losses. We derive the expected total cost per unit time in terms of the decision variables, which are the probability of customer retention $q = 1 - p$ and the service rates μ_1 and μ_2. We also establish a relationship between the service rate and the probability of customer retention. The probability that customers will not be able to join the queue because the maximum queue size has been reached (blocking probability) P_N is given in equation 13 in the previous subsection, and the average reneging rate is

$$R_r = \sum_{n=1}^{N} (n-1)\xi p P_n \tag{18}$$

Where the probability that a customer will leave the queue without entering service (reneging or abandonment probability), p is [**?**]

$$p = \frac{R_r}{\lambda_{eff}} \tag{19}$$

Since some customers leave the queue without actually entering service, the mean final arrival rate of customers that actually enter service is

$$\begin{aligned} \lambda_f &= \lambda_{eff} - R_r \qquad (20) \\ &= \lambda(1 - P_N) - R_r \\ &= \lambda \left[1 - P_N = \prod_{r=1}^{K-1} \left[\frac{\lambda}{(\mu_1 + (r-1)\xi p)} \prod_{l=K}^{N} \left\{ \frac{\lambda}{\mu_2 + (l-1)\xi p} \right\} \right] P_0 \right] - \\ &\sum_{n=1}^{N} (n-1)\xi p P_n \qquad (21) \end{aligned}$$

The expected cost per unit time resulting from the inability of customers to join the queue when its maximum capacity is reached (customer blocking or rejection) or when the customer abandons the queue (customer reneging) is

$$\begin{aligned} L_{TC} &= (\lambda - \lambda_f)C \qquad (22) \\ &= (\lambda P_N + R_r)C \\ &= \left[\lambda \prod_{r=1}^{K-1} \left[\frac{\lambda}{(\mu_1 + (r-1)\xi p)} \prod_{l=K}^{N} \left\{ \frac{\lambda}{\mu_2 + (l-1)\xi p} \right\} \right] P_0 + \sum_{n=1}^{N} (n-1)\xi p P_n \right] C \end{aligned}$$

Where C is the expected loss per customer due to customer rejection or customer reneging. The probability that an arriving customer will not served either because the queue capacity has been reached or because the customer renege from the queue (loss probability), is

$$p_{loss} = \frac{\lambda - \lambda_f}{\lambda} \tag{23}$$

The probability that disatisfied customers will remain in the queue either due to some incentives provided by the service provider or by improvement of the service speed by increasing the service rate (retention probability) is

$$q = 1 - \frac{\lambda \left[1 - \prod_{r=1}^{K-1} \left[\frac{\lambda}{(\mu_1 + (r-1)\xi p)} \prod_{l=K}^{N} \left\{\frac{\lambda}{\mu_2 + (l-1)\xi p}\right\}\right] P_0\right]}{\sum_{n=1}^{N} (n-1)\xi P_n} \tag{24}$$

5 TRANSIENT ANALYSIS

In this section, we perform the transient analysis of the queuing model using Runge-Kutta method of fourth order (RK4). The “ode45” function of MATLAB software is used to find the transient numerical results corresponding to the differential-difference equations of the model as provided in section 3.

In Runge-Kutta method, for solving the differential equation

$$\frac{dy}{dt} = f(t, y)$$
$$y(t_0) = y_0$$

The following formula

$$y_{i+1} = y_i + \left[\frac{k_1 + 2k_2 + 2k_3 + k_4}{6}\right] h$$

where h is the step size and

$$k_1 = hf(t_i, y_i)$$
$$k_2 = hf\left(t_i + \frac{h}{2}, y_i + \frac{1}{2}k_1\right)$$
$$k_3 = hf\left(t_i + \frac{h}{2}, y_i + \frac{1}{2}k_2,\right)$$
$$k_4 = hf\left(t_i + h, y_i + k_3\right)$$

computes an approximate solution.

Considering y as $P_n(t)$, $\frac{dP_n(t)}{dt}$ corresponds to $f(t, y)$. Therefore, the fourth-order Runge-Kutta method can be applied to solve the differential-difference

equations (1)-(4) of the queuing model.
Consider a vector **PA** such that

$$\mathbf{PA}_n(t) = P_n(t); \quad n\text{=0,1,...,}N$$

Similarly, the vectors $\mathbf{AA}_n$, $\mathbf{AB}_n$, $\mathbf{AC}_n$, and $\mathbf{AD}_n$ represent the multivariate version of the intermediate quantities. Vector **AA** corresponds to quantity k_1, therefore,

$$\begin{aligned}
\mathbf{AA}_n &= (\triangle t)\frac{d\mathbf{PA}_n(t)}{dt} \\
&= [-\lambda \mathbf{PA}_0(t) + \mu_1 \mathbf{PA}_1(t)](\triangle t); \qquad n = 0 \\
\text{or} \\
&= [-(\lambda + \mu_1 + (n-1)\xi p)\mathbf{PA}_n(t) + \lambda \mathbf{PA}_{n-1}(t) + \\
&\quad (\mu_1 + n\xi p)\mathbf{PA}_{n+1}(t)]\,(\triangle t); 1 \le n < K \\
\text{or} \\
&= [-\{\lambda + \mu_2 + (n-1)\xi p\}\,\mathbf{PA}_n(t) + \lambda \mathbf{PA}_{n-1}(t) + \\
&\quad (\mu_2 + n\xi p)\mathbf{PA}_{n+1}(t)]\,(\triangle t); K \le n < N \\
\text{or} \\
&= [\lambda \mathbf{PA}_{N-1}(t) - (\mu_2 + (N-1)\xi p)\mathbf{PA}_N(t)]\,(\triangle t); \\
&\quad n = N
\end{aligned}$$

Vector **PB** corresponds to quantity $\left(y + \frac{1}{2}k_1\right)$ such that

$$\mathbf{PB}_n\left(t + \frac{1}{2}\triangle t\right) = \mathbf{PA}_n(t) + \frac{1}{2}\mathbf{AA}_n, \quad n\text{=0,1,...,}N$$

Vector **AB** corresponds to quantity k_2, therefore,

$$\begin{aligned}
\mathbf{AB}_n &= (\triangle t)\frac{d\mathbf{PB}_n\left(t + \frac{1}{2}\triangle t\right)}{dt} \\
&= \left[-\lambda \mathbf{PB}_0\left(t + \frac{1}{2}\triangle t\right) + \mu_1 \mathbf{PB}_1\left(t + \frac{1}{2}\triangle t\right)\right] \\
&\quad \times (\triangle t); n = 0 \\
\text{or} \\
&= \left[-(\lambda + \mu_1 + (n-1)\xi p)\mathbf{PB}_n\left(t + \frac{1}{2}\triangle t\right) + \right. \\
&\quad \lambda \mathbf{PB}_{n-1}\left(t + \frac{1}{2}\triangle t\right) + (\mu_1 + n\xi p)\mathbf{PB}_{n+1} \\
&\quad \left.\left(t + \frac{1}{2}\triangle t\right)\right](\triangle t); \qquad\qquad 1 \le n < K
\end{aligned}$$

or

$$= \left[- \left\{ (\lambda + \mu_2 + (n-1)\xi p) \right\} \mathbf{PB}_n \left(t + \frac{1}{2} \triangle t \right) + \lambda \mathbf{PB}_{n-1} \left(t + \frac{1}{2} \triangle t \right) + (\mu_2 + n\xi p) \mathbf{PB}_{n+1} \left(t + \frac{1}{2} \triangle t \right) \right] (\triangle t); K \leq n < N$$

or

$$= \left[\lambda \mathbf{PB}_{N-1} \left(t + \frac{1}{2} \triangle t \right) - (\mu_2 + (N-1)\xi p) \mathbf{PB}_N \left(t + \frac{1}{2} \triangle t \right) \right] (\triangle t); n = N$$

Vector **PC** corresponds to quantity $\left(y + \frac{1}{2} k_2\right)$ such that

$$\mathbf{PC}_n \left(t + \frac{1}{2} \triangle t \right) = \mathbf{PA}_n(t) + \frac{1}{2} \mathbf{AB}_n, \qquad n=0,1,...,N$$

Vector **AC** corresponds to quantity k_3, therefore,

$$\mathbf{AC}_n = (\triangle t) \frac{d\mathbf{PC}_n \left(t + \frac{1}{2} \triangle t \right)}{dt};$$

$$= \left[-\lambda \mathbf{PC}_0 \left(t + \frac{1}{2} \triangle t \right) + \mu_1 \mathbf{PC}_1 \left(t + \frac{1}{2} \triangle t \right) \right] \times (\triangle t) \quad n = 0$$

or

$$= \left[-(\lambda + \mu_1 + (n-1)\xi p) \mathbf{PC}_n \left(t + \frac{1}{2} \triangle t \right) + \lambda \mathbf{PC}_{n-1} \left(t + \frac{1}{2} \triangle t \right) + (\mu_1 + n\xi p) \mathbf{PC}_{n+1} \left(t + \frac{1}{2} \triangle t \right) \right] (\triangle t); 1 \leq n < K$$

or

$$= \left[- \left\{ \lambda + \mu_2 + (n-1)\xi p \right\} \mathbf{PC}_n \left(t + \frac{1}{2} \triangle t \right) + \lambda \mathbf{PC}_{n-1} \left(t + \frac{1}{2} \triangle t \right) + (\mu_2 + n\xi p) \mathbf{PC}_{n+1} \left(t + \frac{1}{2} \triangle t \right) \right] (\triangle t); K \leq n < N$$

or

$$= \left[\lambda \mathbf{PC}_{N-1} \left(t + \frac{1}{2} \triangle t \right) - (\mu_2 + (N-1)\xi p) \mathbf{PC}_N \left(t + \frac{1}{2} \triangle t \right) \right] (\triangle t) n = N$$

Vector **PD** corresponds to quantity $(y + k_3)$ such that

$$\mathbf{PD}_n(t) = \mathbf{PA}_n(t) + \mathbf{AC}_n, \qquad n{=}0{,}1{,}...{,}N$$

Vector **AD** corresponds to quantity k_4, therefore,

$$\begin{aligned}\mathbf{AD}_n &= (\triangle t)\frac{d\mathbf{PD}_n(t)}{dt}\\ &= [-\lambda \mathbf{PD}_0(t) + \mu_1 \mathbf{PD}_1(t)](\triangle t); n = 0\end{aligned}$$

or

$$\begin{aligned}&= [-(\lambda + \mu_1 + (n-1)\xi p)\mathbf{PD}_n(t) + \lambda \mathbf{PD}_{n-1}(t)+\\ &\quad (\mu_1 + n\xi p)\mathbf{PD}_{n+1}(t)]\,(\triangle t); 1 \leq n < K\end{aligned}$$

or

$$\begin{aligned}&= [-\left\{\lambda + \mu_2 + (n-1)\xi p\right\}\mathbf{PD}_n(t) + \lambda \mathbf{PD}_{n-1}(t)+\\ &\quad (\mu_2 + n\xi p)\mathbf{PD}_{n+1}(t)]\,(\triangle t); K \leq n < N\end{aligned}$$

or

$$\begin{aligned}&= [\lambda \mathbf{PD}_{N-1}(t) - (\mu_2 + (N-1)\xi p)\mathbf{PD}_N(t)]\,(\triangle t);\\ &\quad n = N\end{aligned}$$

Using vectors $\mathbf{AA}_n$, $\mathbf{AB}_n$, $\mathbf{AC}_n$, and $\mathbf{AD}_n$ the probabilities can be computed recursively from the following equation

$$P_n\,(t + \triangle t) = P_n(t) + \frac{1}{6}\left(\mathbf{AA}_n + 2\mathbf{AB}_n + 2\mathbf{AC}_n + \mathbf{AD}_n\right), \quad 0 \leq n \leq N \tag{25}$$

We consider the transient-state performance parameters such as the mean number of customers in the queue, $L_q(t)$, the waiting time $W_q(t)$, the average reneging rate $R_r(t)$ and the average retention rate, $R_R(t)$. The transient-state performance parameters are give below:

$$L_q(t) = \sum_{n=2}^{N}(n-1)P_n(t) \tag{26}$$

$$W_q(t) = \frac{L_q(t)}{\mu[1 - P_0(t) - P_1(t)]}$$

$$where, \mu = \frac{[k \times \mu_1 + (n-k) \times \mu_2]}{n}, k+1 \leq n \leq N \tag{27}$$

$$R_r(t) = \sum_{n=1}^{N}(n-1)\xi p P_n(t)$$

$$R_R(t) = \sum_{n=1}^{N}(n-1)\xi q P_n(t)$$

Figures 6 and 7 show the variation in state probabilities with respect to time. All the probabilities except $P_0(t)$ starts from zero and asymptotically reaches the steady-state. This is due to the initial condition we considered $P_0(t) = 1$. Values of parameters are: $\lambda = 13, \mu_1 = 15, \mu_2 = 18, \xi = 0.05, p = 0.5$ and N=27.

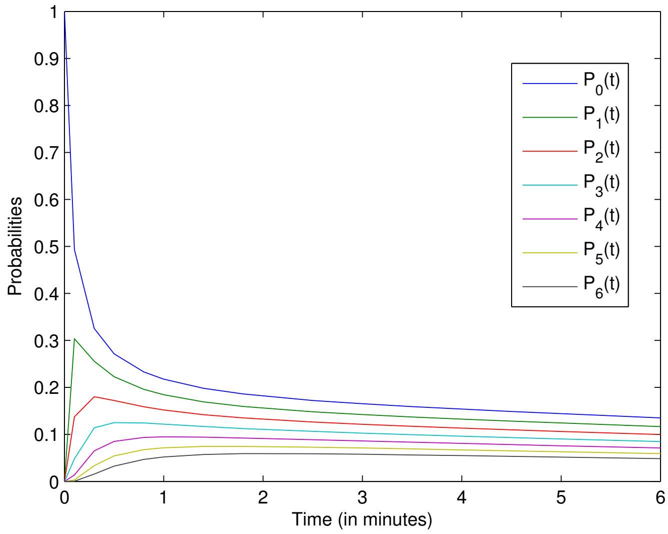

Fig. 6: Probabilities vs Time

In Figures 8 and 9, it is observed that both the expected queue size and expected waiting time in the queue initially increases and then achieved the steady-state. In both the figures, our model has shown higher $L_q(t)$ and $W_q(t)$, but with the progress of time both the performance measures are below the other two models. So, this clearly shows that our model yields better results in the long run for the system.
In Figure 10, the variation in probability of customer rejection is compared with time for the three models. It has been observed that for our proposed model, the probability of customer rejection is way lower than the other two models. This shows that customer in the queuing system with service improvement and retention of impatient customers does not easily leave the system and tends to spend more time in the system which is a positive outlook for any queuing system.

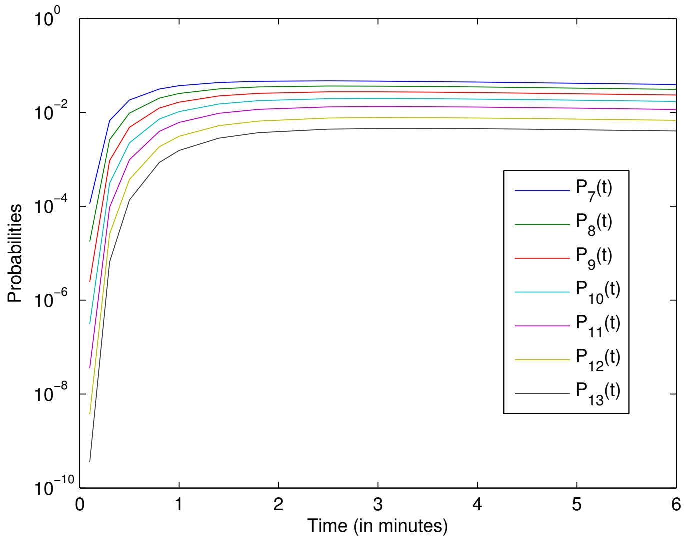

Fig. 7: Probabilities vs Time

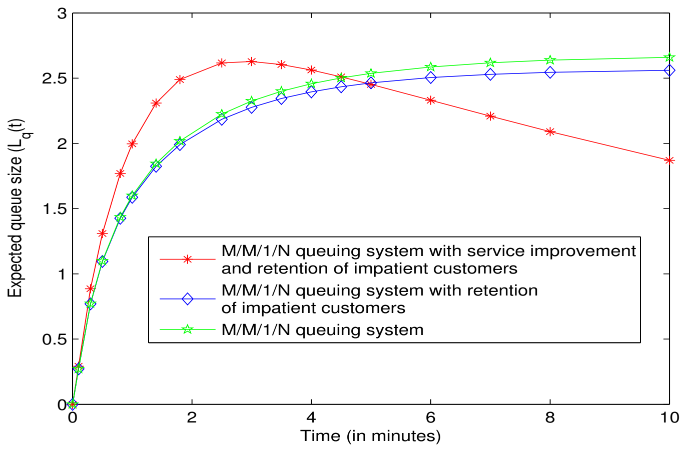

Fig. 8: Variation in expected queue size w.r.t time

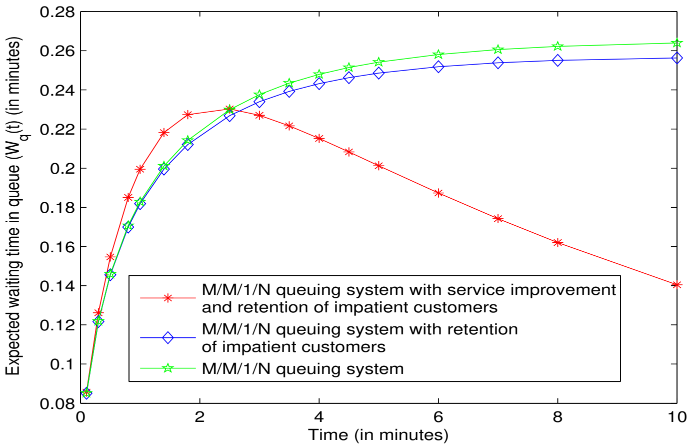

Fig. 9: Variation in expected waiting time in queue w.r.t time

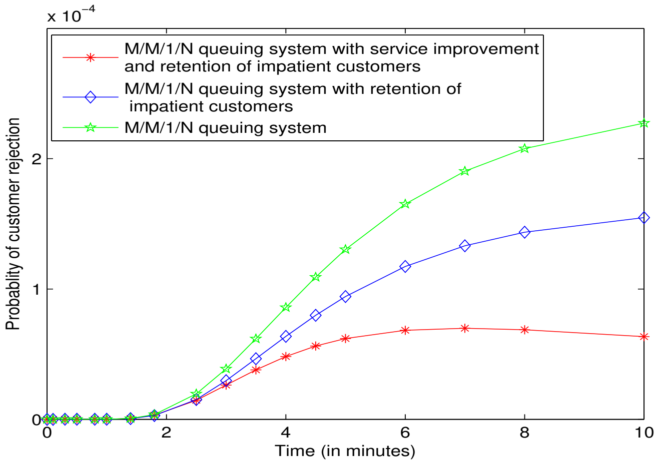

Fig. 10: Variation in probability of customer rejection w.r.t time

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