Controllability of Fractional Linear Systems with Delays

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Abstract—**The main purpose of this paper is to study controllability of linear continuous-time fractional dynamical systems containing both lumped constant delay in state variables and distributed delays in admissible controls. Necessary and sufficient conditions for relative controllability in finite time interval are formulated and proved using theory of linear bounded operators, solution properties of fractional differential equations and results taken directly from linear matrix algebra. The main result of the paper is to show, that global relative controllability of fractional linear systems with different types of delays is equivalent to non-singularity of suitably defined relative controllability matrix.**

Keywords—controllability; linear systems; fractional systems, delayed systems; distributed delays

I. Introduction

Controllability similarly as observability and stability is one of the fundamental concept in mathematical control theory and plays an important role both in traditional and fractional control theory. During recent years controllability of time delay control systems has been considered in many monographs and papers. This has been motivated, on the one hand by the wide range of possible applications in various area of science and engineering and, on the other hand, by the interesting and difficult theoretical problems posed by such systems.

In the theory of dynamical systems we may consider delays both in the state variables or/and in the admissible controls. Moreover, we have generally many kinds of delays, i.e., distributed delays, multiple point delays both constant or time-varying. It is important to note, that in dynamical systems with delays it is necessary to consider two types of states, namely: instantaneous states and complete states.

Controllability of linear systems with different types of delays was considered in many monographs [11], [13], [18], [21], survey papers [19] and [20] and in regular papers [9], [10], [11], [12], [16], [17].

The main purpose of this paper is to study the relative controllability of linear infinite delay dynamical systems containing both multiple lumped time varying delays and distributed delays in the state variables and multiple lumped time varying delays in the in admissible controls.

Controllability is a qualitative property of dynamical control systems and is of particular importance in different, mainly theoretical problems in control theory. Systematic study of controllability was started at the beginning of sixties, when the theory of controllability based on the description in the form of state space for both time-invariant and time-varying linear control systems was worked out [18]. Roughly speaking, controllability generally means, that it is possible to steer dynamical control system from an arbitrary initial state to an arbitrary final state using the set of admissible controls.

In the literature there are many different definitions of controllability, both for linear and nonlinear or semilinear dynamical systems [4], [5], [25], [29]. Controllability concept strongly depends on class of dynamical control systems and on the set of admissible controls, [10], [11], [12], [13], [33]. Therefore, nonlinear or semilinear fractional systems there exist many different necessary and sufficient conditions for global and local controllability [4], [5], [24]. Using theory of difference equations and pure algebraic methods controllability of different discrete time linear fractional control systems was discussed in [10], [11], [17].

The control processes frequently involve different types of delays in state variables or in admissible controls [15]. It should be pointed out, that delay is one of the general phenomenon in real dynamical system, which has a crucial effect on the system properties, as for example on the controllability, observability and stability.

For dynamical systems with delays in control and/or state variables two fundamental concepts of states are considered, namely: finite-dimensional instantaneous or relative state and infinite-dimensional complete or functional state [18], [21]. However, it should be stressed, that relative state does not give full information about trajectory of control system. Hence, it is necessary to introduce at least two different concepts of controllability, namely: relative controllability connected with relative states, and complete controllability connected with complete states.

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On the other hand, fractional order continuous and discrete mathematical models express the behavior of many real process more precisely than integer order ones.

The various types of fractional differential equations have many applications in different fields of technique including for example signal processing, theory of visco-elastic materials [1], [30], supercapacitors [23] filter description and design, circuit theory [13], computer networks, and bioengineering [11].

Recently different controllability problems have been discussed both for linear or nonlinear fractional infinite dimensional control systems defined in Hilbert spaces. Stochastic boundary controllability of nonlinear fractional systems defined in infinite dimensional Hilbert space was considered in paper [27] using methods of stochastic differential equations. Approximation results for linear fractional diffusion wave equation were presented and discussed in paper [22]. Moreover, existence and properties of solution and initial Cauchy problem for abstract infinite dimensional linear differential fractional equation are formulated and discussed in paper [32].

In the present paper we shall study global relative controllability in a given finite time interval for fractional, linear, continuous time dynamical systems with multiple time variable point delays and distributed delay in admissible control.

This is natural generalizations of controllability concepts, which is rather well known in the theory of finite dimensional linear control systems [11], without delays in state variables or admissible control. Using techniques similar to those presented in monographs [18], and [21] and in the series of papers [10], [12], [16] and [17] we shall formulate and prove necessary and sufficient conditions for global relative controllability of fractional control systems in a prescribed time interval.

This paper is organized as follows: section 2 contains mathematical model of linear, stationary fractional stationary dynamical system with multiple time variable point delays in admissible controls. Moreover, in this section basic solution of fractional linear finite dimensional differential equation is presented in compact integral form and its properties are also discussed. In section 3 definition of global relative controllability in a given time interval is recalled. Next, using results and methods taken directly from linear functional analysis [31], global relative controllability problem is mathematically stated and considered. Moreover, using suitably defined relative controllability matrix necessary and sufficient conditions for global relative controllability in a finite time interval are formulated and proved. admissible control. Finally, section 4 contains concluding remarks and proposes some open controllability problems for more general fractional systems.

II. SYSTEM DESCRIPTION

Let us consider linear, fractional, delay dynamical systems containing single lumped constant delay in the state variables and distributed delays in admissible controls, described by the following fractional differential state equation [2], [3], [11], [24], [25].

$$D^{\alpha}x(t) = Ax(t) + Cx(t-h) + \int_{-h}^{0} d_{\tau}B(t,\tau)u(t+\tau) \quad (1)$$

$$t \in [t_0 - h, t_1]$$

with initial complete state

$$x(t) = x_{t_0}(t), \;\; u(t) = u_{t_0}(t) \;\; for \;\; t \in [t_0 - h, t_0] \quad (2)$$

where

$0 < \alpha \le 1$, $D^{\alpha}(t)$ denotes fractional Caputo derivative,

A is $n \times n$ dimensional constant matrix with real coefficients,

admissible controls $u \in U_{ad} = L^2([t_0,t_1],R^p)$ are unconstrained,

$h>0$ is given delay

$x(t) \in R^n$ for $t \in [t_0 - h, t_0]$,

$u(t) \in R^p$ for $t \in [t_0 - h, t_0]$,

$x_{t_0}(t),\; t \in [t_0 - h, t_0]$ is given continuous initial function,

$u_{t_0}(t),\; t \in [t_0 - h, t_0]$ is given initial admissible control

$B(t,\tau)$ is $n \times p$ dimensional matrix continuous in t for fixed τ and of bounded variation in τ on *[-h,0]* for each $t \in [t_0, t_1]$ and continuous from left in τ on the interval *(-h,0)*.

integral term in (1) is in the Lebesque-Stieltjes sense [6], [8], [9] with respect to τ,

symbol dB_{τ} denotes the Lebesque-Stieltjes integration [6], [8], with respect to the variable τ in the matrix function $B(t,\tau)$.

initial data $\{x_{t_0}, u_{t_0}\}$ forms complete state of the fractional delayed system (1) at initial time t_0.

In order to find the solution of fractional differential equation (1) let us use Laplace transform L and let us introduce the following $n \times n$ dimensional matrices [14],

$$X_{\alpha}(t) = L^{-1}\left[\frac{s^{\alpha-1}}{s^{\alpha} - A - Ce^{-s}}\right](t)$$

$$X_{\alpha,\alpha}(t) = t^{1-\alpha}\int_{0}^{t}\frac{(t-s)^{\alpha-2}}{\Gamma(\alpha-1)}X_{\alpha}(s)ds$$

where symbol Γ denotes the Euler gamma function.

III Controllability Conditions

Since in the paper only relative controllability is considered, then let us recall definition of global relative controllability in a given finite time interval.

Definition 1. The system (1) is said to be globally relatively controllable over time interval $[t_0, t_1]$ if for each initial complete state $\{x_{t_0}, u_{t_0}\}$ of and any final relative state $x_1 \in R^n$ there exists an admissible control $u \in L^2([t_0, t_1], R^p)$ such that the solution of equation (1) with initial conditions (2) satisfies final condition $x(t_1) = x_1$.

Solution of equation (1) can be expressed as follows

$$x(t_0, t_1, x_0, u_{t_0}, u) = x_L(t_1, x_{t_0}) + \int_{t_0}^{t_1} (t_1 - s)^{\alpha-1} X_{\alpha,\alpha}(t_1 - s) \left[\int_{-h}^{0} d_\tau B(s,\tau) u(s+\tau) \right] ds \quad (2)$$

where

$$x_L(t_1, x_{t_0}) = X_\alpha(t) x(t_0) + \int_{-h}^{0} (t_1 - s - h)^{\alpha-1} X_{\alpha,\alpha}(t_1 - s - h) x_{t_0}(s) ds$$

Now, using unsymmetric Fubini theorem (see e.g. [6] and [8] for more details) and changing order of integration in the last term we have [2], [19], [26]

$$\begin{aligned} x(t_0, t_1, x_0, u_{t_0}, u) &= x_L(t_1, x_{t_0}) + \int_{-h}^{0} dB_\tau \left[\int_{t_0}^{t_1} (t_1 - s)^{\alpha-1} X_{\alpha,\alpha}(t_1 - s) B(s,\tau) u(s) ds \right] = \\ &= x_L(t_1, x_{t_0}) + \int_{-h}^{0} dB_\tau \left[\int_{\tau}^{t_0} (t_1 - s)^{\alpha-1} X_{\alpha,\alpha}(t_1 - s) B(s-\tau,\tau) u_{t_0}(s) ds \right] + \\ &+ \int_{-h}^{0} dB_\tau \left[\int_{t_0}^{t_1+\tau} (t_1 - s)^{\alpha-1} X_{\alpha,\alpha}(t_1 - s) B(s-\tau,\tau) u(s) ds \right] = \\ &= x_L(t_1, x_{t_0}) + \int_{-h}^{0} dB_\tau \left[\int_{\tau}^{t_0} (t_1 - s)^{\alpha-1} X_{\alpha,\alpha}(t_1 - s) B(s-\tau,\tau) u_{t_0}(s) ds \right] + \\ &+ \int_{t_0}^{t_1} \left[\int_{-h}^{0} (t_1 - s)^{\alpha-1} X_{\alpha,\alpha}(t_1 - s) d_\tau B_{t_1}(s-\tau,\tau) \right] u(s) ds \end{aligned} \quad (3)$$

where

$$B_{t_1}(s,\tau) = \begin{cases} B(s,\tau), & s \le t_1, \\ 0, & s > t_1 \end{cases}$$

The first two terms in formula (3) are depended only on given initial complete state $\{x_{t_0}, u_{t_0}\}$, and in fact do not depend on admissible control $u(t),\ t \ge t_0$. Therefore, in order to separate these terms let us denote

$$q(t_1, t_0, x_{t_0}, u_{t_0}) = x_L(t_1, x_{t_0}) + \int_{-h}^{0} dB_\tau \left[\int_{t_0+\tau}^{t_0} (t_1 - s)^{\alpha-1} X_{\alpha,\alpha}(t_1 - s) B(s-\tau,\tau) u_{t_0}(s) ds \right] \quad (4)$$

Moreover, changing variables in the integral term

$$\left[\int_{-h}^{0} d_\tau B(s,\tau) u(s+\tau) \right]$$

and taking into account the form of solution (3) we obtain

$$x(t_0, t_1, x_0, u_{t_0}, u) = q(t_1, t_0, x_0, u_{t_0}) + \int_{t_0}^{t_1} \left[\int_{-h}^{0} (t_1 - s)^{\alpha-1} X_{\alpha,\alpha}(t_1 - s) d_\tau B_{t_1}(s-\tau,\tau) \right] u(s) ds \quad (5)$$

Now, let us introduce relative controllability operator $C_\alpha(t_1)$ and its adjoint operator $C_\alpha^*(t_1)$

$$C_\alpha(t_1) u = \int_{t_0}^{t_1} \left(\int_{-h}^{0} t_1 - s)^{\alpha-1} X_{\alpha,\alpha}(t_1 - s) d_\tau B_{t_1}(s-\tau,\tau) \right) u(s) ds \quad (6)$$

$$C_\alpha^*(t_1) y = \left(\int_{-h}^{0} t_1 - s)^{\alpha-1} X_{\alpha,\alpha}(t_1 - s) d_\tau B_{t_1}(s-\tau,\tau) \right)^* y \quad (7)$$

Finally, let us define $n \times n$ dimensional relative controllability matrix

$$W(t_0,t_1)=C_\alpha(t_1)C_\alpha^*(t_1)=$$
$$=\int_{t_0}^{t_1}(\int_{-h}^{0}(t_1-s)^{\alpha-1}X_{\alpha,\alpha}(t_1-s)d_\tau B_{t_1}(s-\tau,\tau))\times \quad (8)$$
$$\times(\int_{-h}^{0}(t_1-s)^{\alpha-1}X_{\alpha,\alpha}(t_1-s)^*d_\tau B_{t_1}(s-\tau,\tau))^*ds$$

Using relative controllability matrix it is possible to formulate and prove main result of the paper given the following theorem, which presents necessary and sufficient conditions for global relative controllability in a given time interval.

Theorem 1. The following statements are equivalent

(1) Fractional system (1) is globally relatively controllable over $t\in[t_0,t_1]$.
(2) Relative controllability linear operator $C_\alpha : L^2([t_0,t_1],R^p)\to R^n$ is onto.
(3) Adjoint relative controllability operator $C_\alpha^*: R^n\to L^2([t_0,t_1],R^m)$ is invertible i.e., it is linear "one to one" operator.
(4) The bounded linear operator $C_\alpha C_\alpha^* : R^n\to R^n$ is onto and may be realized by *nxn* nonsingular matrix.

Proof.
In the proof of Theorem 1 relative controllability linear bounded operator C_α and its adjoint operator C_α^* play the important role. Hence, linear functional analysis theory may be applied to prove theorem. More precisely, we shall use methods and results taken directly from theory of linear bounded operators in Hilbert spaces.

First of all, let use, that range of the relative controllability operator C_α is finite dimensional, then operator C_α is a bounded linear operator.

Moreover, as was mentioned before, from the definition 1 and integral formula (8) immediately follows that global relative controllability property is equivalent that relative controllability operator C_α is surjective operator. Hence, equivalence (1) and (2) follows.

From the theory of linear operators follows that surjectivity of the operator C_α implies (see e.g. [8], [31]) that its adjoint linear operator

$$C_\alpha^*: R^n\to L^2([t_0,t_1],R^m)$$

is also linear and bounded operator and moreover it is invertible operator, i.e. "one to one" operator.

Hence, equivalence (2) and (3) follows.

Similarly, from theory of linear bounded operators follows, that invertibility of the selfadjoint operator $C_\alpha C_\alpha^*$ means, that exist inverse bounded linear operator $(C_\alpha C_\alpha^*)^{-1}$ and this is equivalent to surjectivity of the operator C_α. Therefore, for relatively controllable fractional system (1), relative controllability matrix

$$W(t_0,t_1)=C_\alpha C_\alpha^* : R^n\to R^n$$

is invertible i.e., it is full rank matrix. Hence, equivalence (4) and (1) follows.
This statement completes proof of Theorem 1.

Corollary 2. Fractional system (1) with distributed delay in admissible control is globally relatively controllable on time interval $[t_0,t_1]$ if and only if the relative controllability matrix is nonsingular.

Proof. From global relative controllability definition directly follows, that for relatively controllable fractional system (1) the operator relative controllability operator $C_\alpha(t_1)$ is onto. On the other hand by Theorem 1 this is equivalent, that relative controllability matrix $W(t_0,t_1)$ is nonsingular.
For globally relatively controllable fractional system (1) it is possible to find an admissible control, which transforms given initial complete state $\{x_{t_0},u_{t_0}\}$ of and any final relative state $x_1\in R^n$ at time t_1. First of all, let us observe, that since relative controllability matrix $W(t_0,t_1)$ is nonsingular matrix so its inverse $W^{-1}(t_0,t_1)$ is well defined. Therefore, let us define admissible control as follows

$$u^0(t)=C_\alpha^*(t_1)W^{-1}(t_0,t_1)(x_1-x_L(t_1,x_{t_0}))-$$
$$-\int_{-h}^{0}dB_\tau\left[\int_{t_0+\tau}^{t_0}(t_1-s)^{\alpha-1}X_{\alpha,\alpha}(t_1-s)B(s-\tau,\tau)u_{t_0}(s)ds\right]=$$
$$=C_\alpha^*(t_1)W^{-1}(t_0,t_1)(x_1-q(t_0,t_1,x_0,u_{t_0})) \quad (9)$$

where complete initial state and the final relative state vector are chosen arbitrarily.

Inserting admissible control $u^0(t)$ given by equality (9) into solution formula (3) and taking into account equalities (6), (7) and (8) we have

$$x(t_0,t_1,x_0,u_{t_0},u) = q(t_1,t_0,x_0,u_{t_0}) +$$
$$+\int_{t_0}^{t_1}\left[\int_{-h}^{0}(t_1-s)^{\alpha-1}X_{\alpha,\alpha}(t_1-s)d_\tau B_{t_1}(s-\tau,\tau)\right]u^0(s)ds =$$
$$= q(t_1,t_0,x_0,u_{t_0}) +$$
$$+ C_\alpha(t_1)C_\alpha^*(t_1)W^{-1}(t_0,t_1)(x_1 - q(t_1,t_0,x_0,u_{t_0})) =$$
$$= q(t_1,t_0,x_0,u_{t_0}) +$$
$$+ W(t_0,t_1)W^{-1}(t_0,t_1)(x_1 - q(t_1,t_0,x_0,u_{t_0})) = x_1 \tag{10}$$

Thus, the admissible control $u^0(t)$ transfers initial complete state $\{x_{t_0}, u_{t_0}\}$ to final relative state $x_1 \in R^n$ at time t_1.

IV CONCLUSIONS

In this paper linear fractional finite-dimensional stationary dynamical control systems with different types of delays in admissible control are considered. More exactly, single constant delay in state variables and distributed delays in admissible controls are discussed. It is generally assumed, that the mathematical model is represented by linear ordinary fractional differential state equations. Using notations, theorems and methods taken directly from functional analysis and linear controllability theory, necessary and sufficient conditions for global relative controllability in a given finite time interval are formulated and proved.

The main result of this paper is to show and to prove, that global relative controllability of fractional control systems with delays both in state variables and in admissible control is equivalent to non-singularity of suitably defined square relative controllability matrix.

Using suitably defined relative controllability matrix for global relatively controllable systems steering admissible control is proposed, which steers the fractional system from given initial complete state to desired final relative state. Moreover, at the beginning of the paper some remarks and comments on the existing in literature controllability results for different types of linear continuous-time and discrete-time fractional dynamical system are also presented.

It should be pointed out, that using different methods of functional analysis, controllability results presented in this paper may be extended in many different ways both for fractional systems and for standard systems [16], [17], [33] and for fractional systems with constrained admissible controls [9], [10], [11], [12], [26], [27] and [28]. First of all, using relative controllability matrix, relative controllability problems for semilinear, or generally nonlinear fractional control systems with different types of delays not only in admissible controls, but also in the state variables recently have been considered in papers [4], [5].

Second possibility is to formulate and prove necessary and sufficient conditions for relative controllability of fractional control systems with different orders of derivatives, applying methods and concepts proposed in paper [12].

The third direction is to consider infinite dimensional control systems applying functional analysis methods and concepts (see monographs [18], [21], and [32]). Since in this case relative state space is infinite dimensional space, then several additional concepts of controllability should be introduced, namely: approximate absolute controllability and exact absolute controllability, approximate relative controllability and exact relative controllability.

In last few years nonlinear or semilinear fractional control systems have been discussed in the literature e. g., in papers [3], and [4]. However, so far, but only rather little attention reports on the global or local relative controllability for delayed systems were published. It follows from the fact, that for nonlinear or semilinear fractional systems we do not know the exact form of the solution for the nonlinear state equation.

Relative controllability conditions for semilinear fractional systems with dominated linear part are discussed in the papers [24], and [25] under the assumption, that linear part is relatively controllable and the nonlinear part satisfy certain inequality.

Generally, in the case of semilinear or nonlinear fractional control systems different techniques are used. The most popular is the fixed-point technique. For example, it is possible to use Banach fixed point theorem, Schauder fixed point theorem, Schaefer fixed point theorem or Darbou fixed point theorem based on measures of noncompactnes in Banach, spaces [6], [8]. It strongly depends on the form of nonlinear part of the fractional state equation.

Minimum energy control problem for fractional systems, similarly as for standard linear systems is strongly connected with different controllability concepts, (see e.g., [16], [20], [24] for more details). First of all, let us observe, that for relatively controllable linear control system there exists generally many different admissible controls transferring given initial state complete state to the desired final relative state. Therefore, we may ask which, of these possible admissible controls are optimal one according to given a priori criterion.

For quadratic criterion and relatively controllable linear fractional systems (1), solution of this problem can be found using relative controllability matrix. Moreover, minimum energy value may be computed in rather simple form. However, it should be mentioned, that this method requires many additional restrictive assumptions (see monographs [18] and [21] and survey papers [16] and [17] for more details) as for example, that state variables and admissible controls are unbounded in whole time interval.

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